

Chapter 1

Quark Mass Hierarchy and Flavor Mixing Puzzles

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The fact that quarks of the same electric charge possess a mass hierarchy is a big puzzle in particle physics, and it must be highly correlated with the hierarchy of quark flavor mixing. This review article is intended to provide a brief description of some important issues regarding quark masses, flavor mixing and CP violation. A comparison between the salient features of quark and lepton flavor mixing structures is also made.

1. A Brief History of Flavors

In the subatomic world the fundamental building blocks of matter are known as “flavors”, including both quarks and leptons. Fifty years ago, the quark model was born thanks to the seminal work done by Murray Gell-Mann¹ and George Zweig² independently; and it turned out to be a great milestone in the history of particle physics. The phrase “flavor physics” was first coined by Harald Fritzsch and Murray Gell-Mann in 1971, when they were trying different flavors of ice cream at one of the Baskin Robbins stores in Pasadena. Since then quarks and leptons have been *flavored*.

There are totally twelve different flavors within the Standard Model (SM): six quarks and six leptons. Table 1 is an incomplete list of the important discoveries in flavor physics, which can give one a ball-park feeling of a century of developments in particle physics. The SM contains thirteen free flavor parameters in its electroweak sector: three charged-lepton masses, six quark masses, three quark flavor mixing angles and one CP-violating phase. Since the three neutrinos must be massive beyond the SM, one has to introduce seven (or nine) extra free parameters to describe their flavor properties: three neutrino masses, three lepton flavor mixing angles and one

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Table 1. An incomplete list of some important discoveries in the 100-year developments of flavor (quark and lepton) physics.³

	Discoveries of lepton flavors, quark flavors and weak CP violation
1897	electron (Thomson ⁴)
1919	proton (up and down quarks) (Rutherford ⁵)
1932	neutron (up and down quarks) (Chadwick ⁶)
1933	positron (Anderson ⁷)
1937	muon (Neddermeyer and Anderson ⁸)
1947	Kaon (strange quark) (Rochester and Butler ⁹)
1956	electron antineutrino (Cowan <i>et al.</i> ¹⁰)
1962	muon neutrino (Danby <i>et al.</i> ¹¹)
1964	CP violation in <i>s</i> -quark decays (Christenson <i>et al.</i> ¹²)
1974	charm quark (Aubert <i>et al.</i> ¹³ and Abrams <i>et al.</i> ¹⁴)
1975	tau (Perl <i>et al.</i> ¹⁵)
1977	bottom quark (Herb <i>et al.</i> ¹⁶)
1995	top quark (Abe <i>et al.</i> ¹⁷ and Abachi <i>et al.</i> ¹⁸)
2001	tau neutrino (Kodama <i>et al.</i> ¹⁹)
2001	CP violation in <i>b</i> -quark decays (Aubert <i>et al.</i> ²⁰ and Abe <i>et al.</i> ²¹)

(or three) CP-violating phase(s), corresponding to their Dirac (or Majorana) nature^a. So there are at least twenty flavor parameters at low energies. Why is the number of degrees of freedom so big in the flavor sector? What is the fundamental physics behind these parameters? Such puzzles constitute the flavor problems in modern particle physics.

2. Quark Mass Hierarchy

Quarks are always confined inside hadrons, and hence the values of their masses cannot be directly measured. The only way to determine the masses of six quarks is to study their impact on hadron properties based on QCD. Quark mass parameters in the QCD and electroweak Lagrangians depend both on the renormalization scheme adopted to define the theory and on the scale parameter μ — this dependence reflects the fact that a bare quark is surrounded by a cloud of gluons and quark-antiquark pairs. In the limit where all the quark masses vanish, the QCD Lagrangian possesses an $SU(3)_L \times SU(3)_R$ chiral symmetry, under which the left- and right-handed quarks transform independently. The scale of dynamical chiral symmetry breaking (i.e., $\Lambda_\chi \simeq 1$ GeV) can therefore be used to distinguish between

^aIn this connection we have assumed the 3×3 lepton flavor mixing matrix U to be unitary for the sake of simplicity. Whether U is unitary depends on the mechanism of neutrino mass generation.

Table 2. Running quark masses at some typical energy scales in the SM, including the Higgs mass $M_H \simeq 125$ GeV and the vacuum-stability cutoff scale $\Lambda_{\text{VS}} \simeq 4 \times 10^{12}$ GeV.²⁴

μ	m_u (MeV)	m_d (MeV)	m_s (MeV)	m_c (GeV)	m_b (GeV)	m_t (GeV)
$m_c(m_c)$	$2.79^{+0.83}_{-0.82}$	$5.69^{+0.96}_{-0.95}$	116^{+36}_{-24}	$1.29^{+0.05}_{-0.11}$	$5.95^{+0.37}_{-0.15}$	$385.7^{+8.1}_{-7.8}$
2 GeV	$2.4^{+0.7}_{-0.7}$	4.9 ± 0.8	100^{+30}_{-20}	$1.11^{+0.07}_{-0.14}$	$5.06^{+0.29}_{-0.11}$	$322.2^{+5.0}_{-4.9}$
$m_b(m_b)$	$2.02^{+0.60}_{-0.60}$	$4.12^{+0.69}_{-0.68}$	84^{+26}_{-17}	$0.934^{+0.058}_{-0.120}$	$4.19^{+0.18}_{-0.16}$	$261.8^{+3.0}_{-2.9}$
M_W	$1.39^{+0.42}_{-0.41}$	$2.85^{+0.49}_{-0.48}$	58^{+18}_{-12}	$0.645^{+0.043}_{-0.085}$	$2.90^{+0.16}_{-0.06}$	174.2 ± 1.2
M_Z	$1.38^{+0.42}_{-0.41}$	2.82 ± 0.48	57^{+18}_{-12}	$0.638^{+0.043}_{-0.084}$	$2.86^{+0.16}_{-0.06}$	172.1 ± 1.2
M_H	$1.34^{+0.40}_{-0.40}$	$2.74^{+0.47}_{-0.47}$	56^{+17}_{-12}	$0.621^{+0.041}_{-0.082}$	$2.79^{+0.15}_{-0.06}$	$167.0^{+1.2}_{-1.2}$
$m_t(m_t)$	$1.31^{+0.40}_{-0.39}$	2.68 ± 0.46	55^{+17}_{-11}	$0.608^{+0.041}_{-0.080}$	$2.73^{+0.15}_{-0.06}$	163.3 ± 1.1
1 TeV	1.17 ± 0.35	$2.40^{+0.42}_{-0.41}$	49^{+15}_{-10}	$0.543^{+0.037}_{-0.072}$	$2.41^{+0.14}_{-0.05}$	148.1 ± 1.3
Λ_{VS}	$0.61^{+0.19}_{-0.18}$	1.27 ± 0.22	26^{+8}_{-5}	$0.281^{+0.02}_{-0.04}$	$1.16^{+0.07}_{-0.02}$	82.6 ± 1.4
M_q	—	—	—	$1.84^{+0.07}_{-0.13}$	$4.92^{+0.21}_{-0.08}$	172.9 ± 1.1
$m_q(M_q)$	—	—	—	$1.14^{+0.06}_{-0.12}$	$4.07^{+0.18}_{-0.06}$	162.5 ± 1.1

light quarks ($m_q < \Lambda_\chi$) and heavy quarks ($m_q > \Lambda_\chi$).²²

One may make use of the chiral perturbation theory, the lattice gauge theory and QCD sum rules to determine the *current* masses of the three light quarks u , d and s .²³ Their values can be rescaled to $\mu = 2$ GeV in the modified minimal subtraction ($\overline{\text{MS}}$) scheme, as shown in Table 2. On the other hand, the heavy quark effective theory, the lattice gauge theory and QCD sum rules allow us to determine the *pole* masses M_c and M_b of the charm and bottom quarks. The pole mass M_t of the top quark can directly be measured. The relation between the pole mass M_q and the $\overline{\text{MS}}$ running mass $m_q(\mu)$ can be established by taking account of the perturbative QCD corrections.²³ Given the observed mass of the Higgs boson $M_H \simeq 125$ GeV, we have calculated the running masses of six quarks at a number of typical energy scales and listed their values in Table 2,²⁴ where $\Lambda_{\text{VS}} \simeq 4 \times 10^{12}$ GeV denotes the cutoff scale of vacuum stability in the SM. Note that the values of the pole masses M_q and running masses $m_q(M_q)$ themselves, rather than the running masses $m_q(\mu)$ at these mass scales, are given in the last two rows of Table 2 for the sake of comparison. But the pole masses of the three light quarks are not listed, simply because the perturbative QCD calculation is not reliable in that energy region.²⁴

The quark mass values shown in Table 2 indicate the existence of a strong hierarchy either in the (u, c, t) sector or in the (d, s, b) sector. We find that it is instructive to consider the quark mass spectrum at the reference scale $\mu = M_Z$ by adopting the $\overline{\text{MS}}$ scheme. The reason is simply that an extension

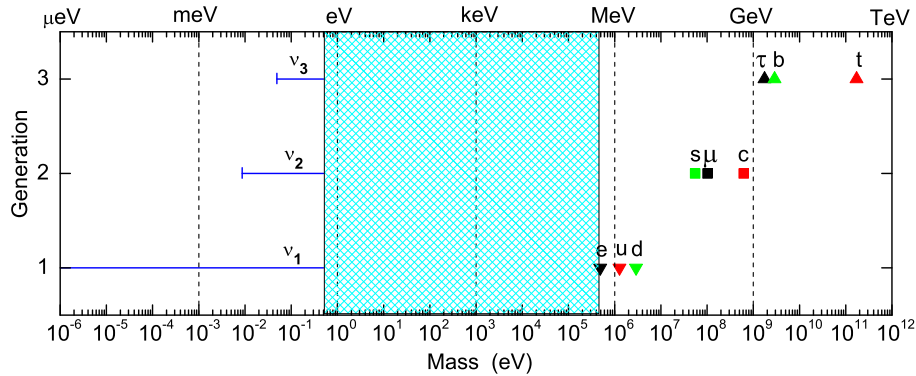


Fig. 1. A schematic illustration of the flavor “hierarchy” and “desert” problems in the SM fermion mass spectrum at the electroweak scale.

of the SM with new physics should be highly necessary far above M_Z , and the strong coupling constant α_s becomes sizable far below M_Z . Quantitatively,

$$\begin{aligned}
 Q_q = +2/3 : \quad & \frac{m_u}{m_c} \simeq \frac{m_c}{m_t} \simeq \lambda^4, \\
 Q_q = -1/3 : \quad & \frac{m_d}{m_s} \simeq \frac{m_s}{m_b} \simeq \lambda^2,
 \end{aligned} \tag{1}$$

hold to an acceptable degree of accuracy, where $\lambda \equiv \sin \theta_C \approx 0.225$ with θ_C being the famous Cabibbo angle of quark flavor mixing.²⁵ The three charged leptons have a similar mass hierarchy.

To be more intuitive, we present a schematic plot for the mass spectrum of six quarks and six leptons at the electroweak scale in Fig. 1,²⁶ where a normal neutrino mass ordering has been assumed. One can see that the span between the neutrino masses m_i and the top-quark mass m_t is at least twelve orders of magnitude. Furthermore, the “desert” between the heaviest neutral fermion (e.g., ν_3) and the lightest charged fermion (i.e., e^-) spans at least six orders of magnitude. Why do the twelve fermions have such a strange mass pattern with the remarkable hierarchy and desert? A convincing answer to this fundamental question remains open.

3. Flavor Mixing Pattern

In a straightforward extension of the SM which allows its three neutrinos to be massive, a nontrivial mismatch between the mass and flavor eigenstates of leptons or quarks arises from the fact that lepton or quark fields can interact with both scalar and gauge fields, leading to the puzzling phenomena of flavor mixing and CP violation. The 3×3 lepton and quark flavor mixing

matrices appearing in the weak charged-current interactions are referred to, respectively, as the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix U ²⁷ and the Cabibbo-Kobayashi-Maskawa (CKM) matrix V :²⁸

$$\begin{aligned} -\mathcal{L}_{cc}^\ell &= \frac{g}{\sqrt{2}} \overline{(e \ \mu \ \tau)_L} \gamma^\mu U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_L W_\mu^- + \text{h.c.} , \\ -\mathcal{L}_{cc}^q &= \frac{g}{\sqrt{2}} \overline{(u \ c \ t)_L} \gamma^\mu V \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L W_\mu^+ + \text{h.c.} , \end{aligned} \quad (2)$$

in which all the fermion fields are the mass eigenstates. By convention, U and V are defined to be associated with W^- and W^+ , respectively. Note that V is unitary as dictated by the SM itself, but whether U is unitary or not depends on the mechanism responsible for the origin of neutrino masses.

In \mathcal{L}_{cc}^ℓ and \mathcal{L}_{cc}^q , the charged leptons and quarks with the same electric charges all have the normal mass hierarchies (namely, $m_e \ll m_\mu \ll m_\tau$, $m_u \ll m_c \ll m_t$ and $m_d \ll m_s \ll m_b$, as shown in Fig. 1 or Table 2). Yet it remains unclear whether the three neutrinos also have a normal mass ordering ($m_1 < m_2 < m_3$) or not. Now that $m_1 < m_2$ has been fixed from the solar neutrino oscillations, the only likely ‘‘abnormal’’ mass ordering is $m_3 < m_1 < m_2$. The neutrino mass ordering is one of the central concerns in flavor physics, and it will be determined in the foreseeable future with the help of either an accelerator-based neutrino oscillation experiment or a reactor-based antineutrino oscillation experiment, or both of them.

Up to now the moduli of nine elements of the CKM matrix V have been determined from current experimental data to a good degree of accuracy:²³

$$|V| = \begin{pmatrix} 0.97427 \pm 0.00015 & 0.22534 \pm 0.00065 & 0.00351_{-0.00014}^{+0.00015} \\ 0.22520 \pm 0.00065 & 0.97344 \pm 0.00016 & 0.0412_{-0.0005}^{+0.0011} \\ 0.00867_{-0.00031}^{+0.00029} & 0.0404_{-0.0005}^{+0.0011} & 0.999146_{-0.000046}^{+0.000021} \end{pmatrix}. \quad (3)$$

We see that V has a clear hierarchy: $|V_{tb}| > |V_{ud}| > |V_{cs}| \gg |V_{us}| > |V_{cd}| \gg |V_{cb}| > |V_{ts}| \gg |V_{td}| > |V_{ub}|$, which must have something to do with the strong hierarchy of quark masses. Fig. 2 illustrates the salient structural features of V , as compared with the more or less ‘‘anarchical’’ structure of the PMNS matrix U . There exist at least two open questions:²⁹ a) Is there any intrinsic relationship between the flavor mixing parameters of leptons and quarks in a certain grand unified theory? b) If yes, does this kind of relationship hold between U and V or between U and V^\dagger (or between U^\dagger and V) which are both associated with W^- (or W^+)?

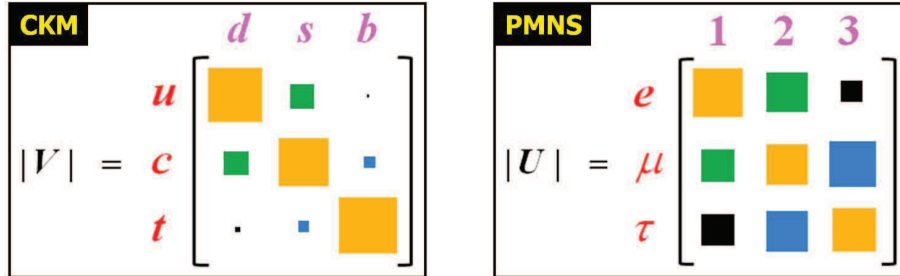


Fig. 2. A schematic illustration of the different flavor mixing structures of six quarks and six leptons at the electroweak scale.

The CKM matrix V can be parametrized in terms of three flavor mixing angles and a nontrivial CP-violating phase in nine different ways.³⁰ Among them, the most popular one is the so-called “standard” parametrization as advocated by the Particle Data Group:²³

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_q} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_q} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_q} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_q} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_q} & c_{13}c_{23} \end{pmatrix}, \quad (4)$$

in which $c_{ij} \equiv \cos \vartheta_{ij}$ and $s_{ij} \equiv \sin \vartheta_{ij}$ (for $ij = 12, 13, 23$) are defined. The present experimental data lead us to

$$\vartheta_{12} = 13.023^\circ \pm 0.038^\circ, \quad \vartheta_{13} = 0.201_{-0.008}^{+0.009}^\circ, \quad \vartheta_{23} = 2.361_{-0.028}^{+0.063}^\circ, \quad (5)$$

and $\delta_q = 69.21_{-4.59}^{+2.55}^\circ$. In comparison, the similar parameters of the PMNS lepton flavor mixing matrix U lie in the following 3σ ranges as obtained from a global analysis of current neutrino oscillation data:³¹

$$\theta_{12} = 30.6^\circ \rightarrow 36.8^\circ, \quad \theta_{13} = 7.6^\circ \rightarrow 9.9^\circ, \quad \theta_{23} = 37.7^\circ \rightarrow 52.3^\circ, \quad (6)$$

and $\delta_\ell = 0^\circ \rightarrow 360^\circ$ provided the neutrino mass ordering is normal (i.e., $m_1 < m_2 < m_3$); or

$$\theta_{12} = 30.6^\circ \rightarrow 36.8^\circ, \quad \theta_{13} = 7.7^\circ \rightarrow 9.9^\circ, \quad \theta_{23} = 38.1^\circ \rightarrow 53.2^\circ, \quad (7)$$

and $\delta_\ell = 0^\circ \rightarrow 360^\circ$ provided the neutrino mass ordering is inverted (i.e., $m_3 < m_1 < m_2$). In either case U exhibits an anarchical pattern as shown in Fig. 2 (left panel). In the literature the possibilities of $\theta_{12} + \vartheta_{12} = 45^\circ$ and $\theta_{23} \pm \vartheta_{23} = 45^\circ$ have been discussed, although such relations depend on both the chosen parametrization and the chosen energy scale.³²

It is worth mentioning the off-diagonal asymmetries of the CKM matrix V in modulus,³³ which provide another measure of the structure of V about

its V_{ud} - V_{cs} - V_{tb} and V_{ub} - V_{cs} - V_{td} axes, respectively:

$$\begin{aligned}\Delta_L^q &\equiv |V_{us}|^2 - |V_{cd}|^2 = |V_{cb}|^2 - |V_{ts}|^2 = |V_{td}|^2 - |V_{ub}|^2 \simeq A^2 \lambda^6 (1 - 2\rho) , \\ \Delta_R^q &\equiv |V_{us}|^2 - |V_{cb}|^2 = |V_{cd}|^2 - |V_{ts}|^2 = |V_{tb}|^2 - |V_{ud}|^2 \simeq \lambda^2 ,\end{aligned}\quad (8)$$

where $A \simeq 0.811$, $\lambda \simeq 0.225$ and $\rho \simeq 0.134$ denote the so-called Wolfenstein parameters.³⁴ It becomes obvious that $\Delta_L^q \simeq 6.3 \times 10^{-5}$ and $\Delta_R^q \simeq 5.1 \times 10^{-2}$ hold, implying that the CKM matrix V is symmetric about its V_{ud} - V_{cs} - V_{tb} axis to a high degree of accuracy. In comparison, the PMNS matrix U is not that symmetric about its either axis, but it may possess an approximate or partial μ - τ permutation symmetry,³⁵ i.e., $|U_{\mu i}| \simeq |U_{\tau i}|$ (for $i = 1, 2, 3$). Such an interesting lepton flavor mixing structure at low energies might originate, via the renormalization-group running effects, from a superhigh-energy PMNS matrix with the exact μ - τ symmetry.³⁶

4. The Unitarity Triangles

Thanks to the six orthogonality relations of the unitary CKM matrix V , one may define six unitarity triangles in the complex plane:

$$\begin{aligned}\Delta_\alpha : & \quad V_{\beta d} V_{\gamma d}^* + V_{\beta s} V_{\gamma s}^* + V_{\beta b} V_{\gamma b}^* = 0 , \\ \Delta_i : & \quad V_{uj} V_{uk}^* + V_{cj} V_{ck}^* + V_{tj} V_{tk}^* = 0 ,\end{aligned}\quad (9)$$

where α, β and γ co-cyclically run over the up-type quarks u, c and t , while i, j and k co-cyclically run over the down-type quarks d, s and b . The inner angles of triangles Δ_α and Δ_i are universally defined as

$$\Phi_{\alpha i} \equiv \arg \left(-\frac{V_{\beta j} V_{\gamma j}^*}{V_{\beta k} V_{\gamma k}^*} \right) = \arg \left(-\frac{V_{\beta j} V_{\beta k}^*}{V_{\gamma j} V_{\gamma k}^*} \right) ,\quad (10)$$

where the Greek and Latin subscripts keep their separate co-cyclical running. So Δ_α and Δ_i share a common inner angle $\Phi_{\alpha i}$, as shown in Fig. 3.

Let us proceed to define the Jarlskog invariant of CP violation \mathcal{J}_q for the CKM matrix V through the equation³⁷

$$\text{Im} (V_{\alpha i} V_{\beta j} V_{\alpha j}^* V_{\beta i}^*) = \mathcal{J}_q \sum_{\gamma} \epsilon_{\alpha\beta\gamma} \sum_k \epsilon_{ijk} ,\quad (11)$$

where the relevant Greek and Latin subscripts run over (u, c, t) and (d, s, b) , respectively. Given the standard (or Wolfenstein) parametrization of V ,

$$\mathcal{J}_q = c_{12} s_{12} c_{13}^2 s_{13} c_{23} s_{23} \sin \delta_q \simeq A^2 \lambda^6 \eta ,\quad (12)$$

where $\eta \simeq 0.354$ is the fourth Wolfenstein parameter.³⁴ We are therefore left with $\mathcal{J}_q \simeq 3.0 \times 10^{-5}$, comparable with Δ_L^q in magnitude. The six CKM

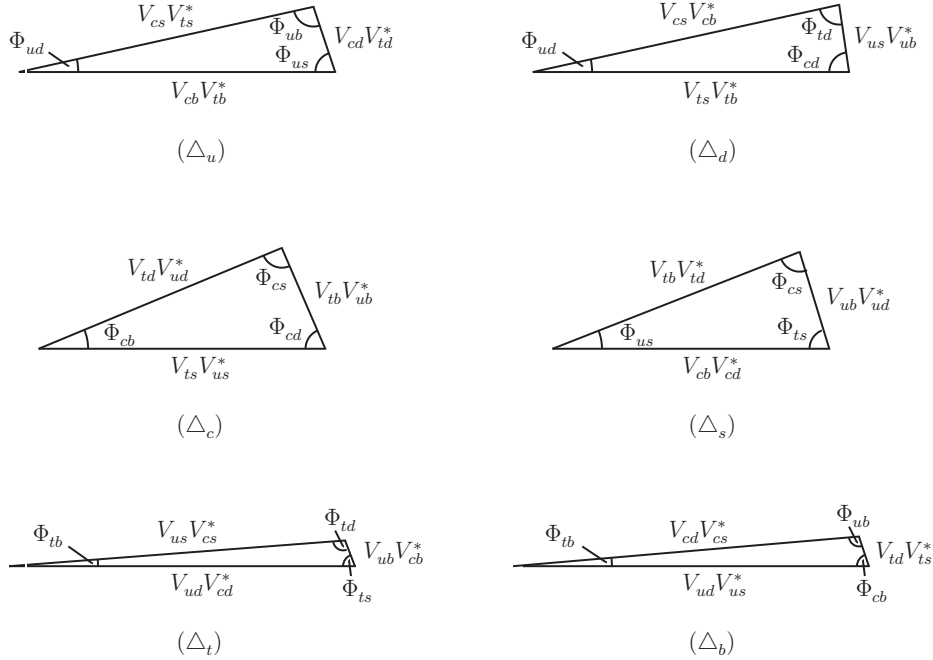


Fig. 3. A schematic illustration of the six CKM unitarity triangles in the complex plane, where each triangle is named by the flavor index that does not manifest itself in the sides.

unitarity triangles have the same area, equal to $\mathcal{J}_q/2$. If Δ_L^q or Δ_R^q were vanishing, there would be the congruence between two unitarity triangles:³⁸

$$\begin{aligned} \Delta_L^q = 0 &\implies \Delta_u \cong \Delta_d, \quad \Delta_c \cong \Delta_s, \quad \Delta_t \cong \Delta_b, \\ \Delta_R^q = 0 &\implies \Delta_u \cong \Delta_b, \quad \Delta_c \cong \Delta_s, \quad \Delta_t \cong \Delta_d. \end{aligned} \quad (13)$$

Fig. 3 clearly shows that $\Delta_L^q \simeq 0$ is actually a rather good approximation.

Note that triangle Δ_s has been well studied in B -meson physics, and its three inner angles are usually denoted as $\alpha = \Phi_{cs}$, $\beta = \Phi_{us}$ and $\gamma = \Phi_{ts}$. A very striking result is $\alpha = 89.0^{+4.4}_{-4.2}^\circ$ as reported by the Particle Data Group,²³ implying that triangles Δ_s and Δ_c are almost the *right* triangles. In fact, $\alpha = 90^\circ$ leads us to the parameter correlation $\mathcal{J}_q = |V_{ud}| \cdot |V_{ub}| \cdot |V_{td}| \cdot |V_{tb}|$, or equivalently $\cos \delta_q = \sin \vartheta_{13}/(\tan \vartheta_{12} \tan \vartheta_{23})$ or $\eta \simeq \sqrt{\rho(1-\rho)}$. If the CKM matrix V is parametrized as³⁹

$$V = \begin{pmatrix} s_u s_d c_h + c_u c_d e^{-i\phi} & s_u c_d c_h - c_u s_d e^{-i\phi} & s_u s_h \\ c_u s_d c_h - s_u c_d e^{-i\phi} & c_u c_d c_h + s_u s_d e^{-i\phi} & c_u s_h \\ -s_d s_h & -c_d s_h & c_h \end{pmatrix}, \quad (14)$$

where $c_x \equiv \cos \vartheta_x$ and $s_x \equiv \sin \vartheta_x$ with $x = u$ (up), d (down) or h (heavy),

then one can easily obtain the following relationship:⁴⁰

$$\frac{\sin \alpha}{\sin \phi} \simeq 1 - \tan \vartheta_u \tan \vartheta_d \cos \vartheta_h \cos \phi - \frac{1}{2} \tan^2 \vartheta_u \tan^2 \vartheta_d \cos^2 \vartheta_h, \quad (15)$$

where the higher-order terms of $\tan \vartheta_u$ and $\tan \vartheta_d$ have been omitted. So $\alpha \simeq \phi$ holds to an excellent degree of accuracy. Taking account of

$$\begin{aligned} \vartheta_u &= \arctan \left(\left| \frac{V_{ub}}{V_{cb}} \right| \right) \simeq 4.87^\circ, \\ \vartheta_d &= \arctan \left(\left| \frac{V_{td}}{V_{ts}} \right| \right) \simeq 12.11^\circ, \\ \vartheta_h &= \arcsin \left(\sqrt{|V_{ub}|^2 + |V_{cb}|^2} \right) \simeq 2.37^\circ, \end{aligned} \quad (16)$$

we obtain $\alpha \simeq 88.95^\circ$ from $\phi = 90^\circ$ according to Eq. (15). The numerical result $\phi - \alpha \simeq 1.05^\circ$ is extremely interesting in the sense that our current experimental data strongly hint at $\phi = 90^\circ$ either at the electroweak scale or at a superhigh-energy scale. Such a conclusion holds because it has been proved that both ϕ and α are completely insensitive to the renormalization-group running effects.^{40,42} Note also that ϕ essentially measures the phase difference between the up-type quark mass matrix and the down-type quark mass matrix, and thus $\phi = 90^\circ$ might have very profound meaning with respect to the origin of CP violation. Some authors have also discussed the possibilities of geometrical, maximal or minimal CP violation⁴¹ in the quark sector.

Finally let us make a brief comment on the so-called CKM phase matrix, whose elements are just the nine inner angles of the CKM unitarity triangles as defined in Eq. (10):^{42,43}

$$\Phi = \begin{pmatrix} \Phi_{ud} & \Phi_{us} & \Phi_{ub} \\ \Phi_{cd} & \Phi_{cs} & \Phi_{cb} \\ \Phi_{td} & \Phi_{ts} & \Phi_{tb} \end{pmatrix} \simeq \begin{pmatrix} 1.05^\circ & 21.38^\circ & 157.57^\circ \\ 68.65^\circ & 88.95^\circ & 22.4^\circ \\ 110.3^\circ & 69.67^\circ & 0.034^\circ \end{pmatrix}. \quad (17)$$

Each row or column of Φ corresponds to an explicit unitarity triangle as illustrated in Fig. 3, and thus its three matrix elements must satisfy the following six sum rules:

$$\sum_{\alpha} \Phi_{\alpha i} = \sum_i \Phi_{\alpha i} = 180^\circ. \quad (18)$$

So one may similarly define two off-diagonal asymmetries of Φ about its two axes.⁴² If one of the two asymmetries were vanishing, we would be left with a result which is analogous to the one in Eq. (13). Of course, one may easily extend the same language to describe the PMNS phase matrix and discuss its evolution with the energy scales.⁴⁴

5. Two Important Limits

Now let us speculate whether the observed pattern of the CKM matrix V can be partly understood in some reasonable limits of quark masses. This idea is more or less motivated by two useful working symmetries in understanding the strong interactions of quarks and hadrons by means of QCD or an effective field theory based on QCD:⁴⁵ the chiral quark symmetry (i.e., $m_u, m_d, m_s \rightarrow 0$) and the heavy quark symmetry (i.e., $m_c, m_b, m_t \rightarrow \infty$). The reason for the usefulness of these two symmetries is simply that the masses of the light quarks are far below the typical QCD scale $\Lambda_{\text{QCD}} \sim 0.2$ GeV, whereas the masses of the heavy quarks are far above it. Because the elements of V are dimensionless and their magnitudes lie in the range of 0 to 1, they can only depend on the mass ratios of the lighter quarks to the heavier quarks. The mass limits corresponding to the chiral and heavy quark symmetries are therefore equivalent to setting the relevant mass ratios to zero, and they are possible to help reveal a part of the salient features of V . In this spirit, some preliminary attempts have been made to look at the quark flavor mixing pattern in the $m_u, m_d \rightarrow 0$ or $m_t, m_b \rightarrow \infty$ limits.⁴⁶

We shall show that it is possible to gain an insight into the observed pattern of quark flavor mixing in the chiral and heavy quark mass limits. This model-independent access to the underlying quark flavor structure can explain why $|V_{us}| \simeq |V_{cd}|$ and $|V_{cb}| \simeq |V_{ts}|$ hold to a good degree of accuracy, why $|V_{cd}/V_{td}| \simeq |V_{cs}/V_{ts}| \simeq |V_{tb}/V_{cb}|$ is a reasonable approximation, and why $|V_{ub}/V_{cb}|$ should be smaller than $|V_{td}/V_{ts}|$. Furthermore, the empirical relations $|V_{ub}/V_{cb}| \sim \sqrt{m_u/m_c}$ and $|V_{td}/V_{ts}| \simeq \sqrt{m_d/m_s}$ can be reasonably conjectured in the heavy quark mass limits.⁴⁷

Let us begin with the CKM matrix $V = O_u^\dagger O_d$ with O_u and O_d being the unitary transformations responsible for the diagonalizations of the up- and down-type quark mass matrices in the flavor basis. Namely,

$$\begin{aligned} O_u^\dagger H_u O_u &= O_u^\dagger M_u M_u^\dagger O_u = \text{Diag} \{m_u^2, m_c^2, m_t^2\}, \\ O_d^\dagger H_d O_d &= O_d^\dagger M_d M_d^\dagger O_d = \text{Diag} \{m_d^2, m_s^2, m_b^2\}, \end{aligned} \quad (19)$$

where H_u and H_d are defined to be Hermitian. To be more explicit, the nine matrix elements of V read

$$V_{\alpha i} = \sum_{k=1}^3 (O_u)_{k\alpha}^* (O_d)_{ki}, \quad (20)$$

where α and i run over (u, c, t) and (d, s, b) , respectively. In general, the mass limit $m_u \rightarrow 0$ (or $m_d \rightarrow 0$) does not correspond to a unique form of H_u (or H_d). The reason is simply that the form of a fermion mass matrix

is always basis-dependent. Without loss of any generality, one may choose a particular flavor basis such that H_u and H_d can be written as

$$\begin{aligned}\lim_{m_u \rightarrow 0} H_u &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}, \\ \lim_{m_d \rightarrow 0} H_d &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix},\end{aligned}\quad (21)$$

in which “ \times ” denotes an arbitrary nonzero element. Note that Eq. (21) is the result of a basis choice instead of an assumption.⁴⁷ When the mass of a given quark goes to infinity, we argue that it becomes decoupled from the masses of other quarks. In this case one may also choose a specific flavor basis where H_u and H_d can be written as

$$\begin{aligned}\lim_{m_t \rightarrow \infty} H_u &= \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \infty \end{pmatrix}, \\ \lim_{m_b \rightarrow \infty} H_d &= \begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \infty \end{pmatrix}.\end{aligned}\quad (22)$$

In other words, the 3×3 Hermitian matrices H_u and H_d can be simplified to the effective 2×2 Hermitian matrices in either the chiral quark mass limit or the heavy quark mass limit. In view of the fact that $m_u \ll m_c \ll m_t$ and $m_d \ll m_s \ll m_b$ hold at an arbitrary energy scale, as shown in Table 2, we believe that Eqs. (21) and (22) are phenomenologically reasonable and can help explain some of the observed properties of quark flavor mixing in a model-independent way. Let us go into details.

(1) *Why $|V_{us}| \simeq |V_{cd}|$ and $|V_{cb}| \simeq |V_{ts}|$ hold?* — A glance at Eq. (3) tells us that $|V_{us}| \simeq |V_{cd}|$ is an excellent approximation. It can be well understood in the heavy quark mass limits, where Hermitian H_u and H_d may take the form of Eq. (22). In this case the unitary matrices O_u and O_d used to diagonalize H_u and H_d can be expressed as

$$\begin{aligned}\lim_{m_t \rightarrow \infty} O_u &= P_{12} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \lim_{m_b \rightarrow \infty} O_d &= P'_{12} \begin{pmatrix} c'_{12} & s'_{12} & 0 \\ -s'_{12} & c'_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix},\end{aligned}\quad (23)$$

where $c_{12}^{(\prime)} \equiv \cos \vartheta_{12}^{(\prime)}$, $s_{12}^{(\prime)} \equiv \sin \vartheta_{12}^{(\prime)}$, and $P_{12}^{(\prime)} = \text{Diag} \{ e^{i\phi_{12}^{(\prime)}}, 1, 1 \}$. Therefore, we immediately arrive at

$$|V_{us}| = |c_{12}s'_{12} - s_{12}c'_{12}e^{i\Delta_{12}}| = |V_{cd}| \quad (24)$$

in the $m_t \rightarrow \infty$ and $m_b \rightarrow \infty$ limits, where $\Delta_{12} \equiv \phi'_{12} - \phi_{12}$ denotes the nontrivial phase difference between the up- and down-quark sectors. Since $m_u/m_c \sim m_c/m_t \sim \lambda^4$ and $m_d/m_s \sim m_s/m_b \sim \lambda^2$ hold, the mass limits taken above are surely a good approximation. So the approximate equality $|V_{us}| \simeq |V_{cd}|$ is naturally attributed to the fact that both $m_t \gg m_u, m_c$ and $m_b \gg m_d, m_s$ hold ^b.

One may similarly consider the chiral quark mass limits $m_u \rightarrow 0$ and $m_d \rightarrow 0$ so as to understand why $|V_{ts}| \simeq |V_{cb}|$ holds. Eq. (21) leads us to

$$\begin{aligned} \lim_{m_u \rightarrow 0} O_u &= P_{23} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \\ \lim_{m_d \rightarrow 0} O_d &= P'_{23} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c'_{23} & s'_{23} \\ 0 & -s'_{23} & c'_{23} \end{pmatrix}, \end{aligned} \quad (25)$$

in which $c_{23}^{(\prime)} \equiv \cos \vartheta_{23}^{(\prime)}$, $s_{23}^{(\prime)} \equiv \sin \vartheta_{23}^{(\prime)}$, and $P_{23}^{(\prime)} = \text{Diag} \{ 1, 1, e^{i\phi_{23}^{(\prime)}} \}$. We are therefore left with

$$|V_{cb}| = |c_{23}s'_{23} - s_{23}c'_{23}e^{i\Delta_{23}}| = |V_{ts}| \quad (26)$$

in the $m_u \rightarrow 0$ and $m_d \rightarrow 0$ limits, where $\Delta_{23} \equiv \phi'_{23} - \phi_{23}$ stands for the nontrivial phase difference between the up- and down-quark sectors. This model-independent result is also in good agreement with the experimental data $|V_{cb}| \simeq |V_{ts}|$ as given in Eq. (3). Namely, the approximate equality $|V_{cb}| \simeq |V_{ts}|$ is a natural consequence of $m_u \ll m_c, m_t$ and $m_d \ll m_s, m_b$ in no need of any specific assumptions ^c.

(2) *Why $|V_{cd}/V_{td}| \simeq |V_{cs}/V_{ts}| \simeq |V_{tb}/V_{cb}|$ holds?* — Given the magnitudes of the CKM matrix elements in Eq. (3), it is easy to get $|V_{cd}/V_{td}| \simeq 26.0$, $|V_{cs}/V_{ts}| \simeq 24.1$ and $|V_{tb}/V_{cb}| \simeq 24.3$. Thus $|V_{cd}/V_{td}| \simeq |V_{cs}/V_{ts}| \simeq |V_{tb}/V_{cb}|$ holds as a reasonably good approximation. We find that such an approximate relation becomes exact in the mass limits $m_u \rightarrow 0$ and $m_b \rightarrow \infty$. To be much

^bQuantitatively, $|V_{us}| \simeq |V_{cd}| \simeq \lambda$ holds. Hence $s_{12} \simeq \sqrt{m_u/m_c} \simeq \lambda^2$ and $s'_{12} \simeq \sqrt{m_d/m_s} \simeq \lambda$ are often conjectured and can easily be derived from some ansätze of quark mass matrices.⁴⁸

^cIt is possible to obtain the quantitative relationship $|V_{cb}| \simeq |V_{ts}| \simeq \lambda^2$ through $s_{23} \simeq m_c/m_t \simeq \lambda^4$ and $s'_{23} \simeq m_s/m_b \simeq \lambda^2$ from a number of ansätze of quark mass matrices.³⁸

more explicit, we arrive at

$$V = \lim_{m_u \rightarrow 0} O_u^\dagger \lim_{m_b \rightarrow \infty} O_d = P'_{12} \begin{pmatrix} c'_{12} & s'_{12} & 0 \\ -c_{23}s'_{12} & c_{23}c'_{12} & -s_{23} \\ -s_{23}s'_{12} & s_{23}c'_{12} & c_{23} \end{pmatrix} P_{23}^\dagger, \quad (27)$$

where Eqs. (23) and (25) have been used. Therefore,

$$\left| \frac{V_{cd}}{V_{td}} \right| = \left| \frac{V_{cs}}{V_{ts}} \right| = \left| \frac{V_{tb}}{V_{cb}} \right| = |\cot \vartheta_{23}| \quad (28)$$

holds in the chosen quark mass limits, which assure the smallest CKM matrix element V_{ub} to vanish. This simple result is essentially consistent with the experimental data if $\vartheta_{23} \simeq 2.35^\circ$ is taken^d. Note that the quark mass limits $m_t \rightarrow \infty$ and $m_d \rightarrow 0$ are less favored because they predict both $|V_{td}| = 0$ and $|V_{us}/V_{ub}| = |V_{cs}/V_{cb}| = |V_{tb}/V_{ts}|$, which are in conflict with current experimental data. In particular, the limit $|V_{ub}| = 0$ is apparently closer to reality than the limit $|V_{td}| = 0$. But why V_{ub} is smaller in magnitude than all the other CKM matrix elements remains a puzzle, since it is difficult for us to judge that the quark mass limits $m_u \rightarrow \infty$ and $m_b \rightarrow 0$ should make more sense than the quark mass limits $m_t \rightarrow \infty$ and $m_d \rightarrow 0$ from a phenomenological point of view. The experimental data in Eq. (3) indicate $|V_{td}| \gtrsim 2|V_{ub}|$ and $|V_{ts}| \simeq |V_{cb}|$. So a comparison between the ratios $|V_{ub}/V_{cb}|$ and $|V_{td}/V_{ts}|$ might be able to tell us an acceptable reason for $|V_{td}| > |V_{ub}|$.

(3) *Why $|V_{ub}/V_{cb}|$ is smaller than $|V_{td}/V_{ts}|$?* — Given Eqs. (21), (22) and (23), we can calculate the ratios $|V_{ub}/V_{cb}|$ and $|V_{td}/V_{ts}|$ in the respective heavy quark mass limits:⁴⁷

$$\begin{aligned} \lim_{m_b \rightarrow \infty} \left| \frac{V_{ub}}{V_{cb}} \right| &= \left| \frac{(O_u)_{3u}}{(O_u)_{3c}} \right|, \\ \lim_{m_t \rightarrow \infty} \left| \frac{V_{td}}{V_{ts}} \right| &= \left| \frac{(O_d)_{3d}}{(O_d)_{3s}} \right|. \end{aligned} \quad (29)$$

This result is quite nontrivial in the sense that $|V_{ub}/V_{cb}|$ turns out to be independent of the mass ratios of three down-type quarks in the $m_b \rightarrow \infty$ limit, and $|V_{td}/V_{ts}|$ has nothing to do with the mass ratios of three up-type quarks in the $m_t \rightarrow \infty$ limit. In particular, the flavor indices showing up on the right-hand side of Eq. (29) is rather suggestive: $|V_{ub}/V_{cb}|$ is relevant to u and c quarks, and $|V_{td}/V_{ts}|$ depends on d and s quarks. We are therefore encouraged to conjecture that $|V_{ub}/V_{cb}|$ (or $|V_{td}/V_{ts}|$) should be a simple function of the mass ratio m_u/m_c (or m_d/m_s) in the $m_t \rightarrow \infty$ (or $m_b \rightarrow \infty$)

^dThis numerical estimate implies $\tan \vartheta_{23} \simeq \lambda^2 \simeq \sqrt{m_c/m_t}$, which can easily be derived from the Fritzsch ansatz of quark mass matrices.⁴⁹

limit. If the values of m_u , m_d , m_s and m_c in Table 2 are taken into account, the simplest phenomenological conjectures should be

$$\begin{aligned} \lim_{m_b \rightarrow \infty} \left| \frac{V_{ub}}{V_{cb}} \right| &\simeq c_1 \sqrt{\frac{m_u}{m_c}}, \\ \lim_{m_t \rightarrow \infty} \left| \frac{V_{td}}{V_{ts}} \right| &\simeq c_2 \sqrt{\frac{m_d}{m_s}}, \end{aligned} \quad (30)$$

where c_1 and c_2 are the coefficients of $\mathcal{O}(1)$. In view of $\sqrt{m_u/m_c} \simeq \lambda^2$ and $\sqrt{m_d/m_s} \simeq \lambda$, we expect that $|V_{ub}/V_{cb}|$ is naturally smaller than $|V_{td}/V_{ts}|$ in the heavy quark mass limits. Taking $c_1 = 2$ and $c_2 = 1$ for example, we obtain $|V_{ub}/V_{cb}| \simeq 0.093$ and $|V_{td}/V_{ts}| \simeq 0.222$ from Eq. (30), consistent with current data $|V_{ub}/V_{cb}| \simeq 0.085$ and $|V_{td}/V_{ts}| \simeq 0.214$ in Eq. (3). Given $m_t \simeq 172$ GeV and $m_b \simeq 2.9$ GeV at M_Z , one may argue that $m_t \rightarrow \infty$ is a much better limit and thus the relation $|V_{td}/V_{ts}| \simeq \sqrt{m_d/m_s}$ has a good chance to be true. In comparison, $|V_{ub}/V_{cb}| \simeq 2\sqrt{m_u/m_c}$ suffers from much bigger uncertainties associated with the values of m_u and m_c , and even its coefficient “2” is questionable.

6. On the Texture Zeros

In the lack of a quantitatively convincing flavor theory, one has to make use of possible flavor symmetries or assume possible texture zeros to reduce the number of free parameters associated with the fermion mass matrices, so as to achieve some phenomenological predictions for flavor mixing and CP violation. Note that the texture zeros of a given fermion mass matrix mean that the corresponding matrix elements are either exactly vanishing or sufficiently suppressed as compared with their neighboring counterparts. There are usually two types of texture zeros:

- They may just originate from a proper choice of the flavor basis, and thus have no definite physical meaning;
- They originate as a natural or contrived consequence of an underlying discrete or continuous flavor symmetry.

A typical example of this kind is the famous Fritzsch mass matrices with six texture zeros,^{49 e} in which three of them come from the basis transformation and the others arise from either a phenomenological assumption or a flavor model (e.g., based on the Froggatt-Nielson mechanism⁵⁰). Such zeros allow one to establish a few simple and testable relations between flavor mixing

^eGiven a Hermitian or symmetric mass matrix, a pair of off-diagonal texture zeros have been counted as one zero in the literature.

angles and fermion mass ratios. If such relations are in good agreement with the relevant experimental data, they may have a good chance to be close to the truth — namely, the same or similar relations should be predicted by a more fundamental flavor model with much fewer free parameters. Hence a study of possible texture zeros of fermion mass matrices *does* make some sense to get useful hints about flavor dynamics that is responsible for the generation of fermion masses and the origin of CP violation.

The original six-zero Fritzsch quark mass matrices was ruled out in the late 1980's, because it failed in making the smallness of V_{cb} compatible with the largeness of m_t . A straightforward extension of the Fritzsch ansatz with five or four texture zeros have been discussed by a number of authors.³⁸ Given current experimental data on quark flavor mixing and CP violation, it is found that only the following five five-zero Hermitian textures of quark mass matrices are still allowed at the 2σ level:⁵¹

$$M_u = \begin{pmatrix} 0 \times 0 \\ \times \times \times \\ 0 \times \times \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 \times 0 \\ \times \times 0 \\ 0 \ 0 \times \end{pmatrix}; \quad (31)$$

or

$$M_u = \begin{pmatrix} 0 \times 0 \\ \times \times 0 \\ 0 \ 0 \times \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 \times 0 \\ \times \times \times \\ 0 \times \times \end{pmatrix}; \quad (32)$$

or

$$M_u = \begin{pmatrix} 0 \times 0 \\ \times 0 \times \\ 0 \times \times \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 \times 0 \\ \times \times \times \\ 0 \times \times \end{pmatrix}; \quad (33)$$

or

$$M_u = \begin{pmatrix} 0 \ 0 \times \\ 0 \times \times \\ \times \times \times \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 \times 0 \\ \times \times 0 \\ 0 \ 0 \times \end{pmatrix}; \quad (34)$$

or

$$M_u = \begin{pmatrix} 0 \ 0 \times \\ 0 \times 0 \\ \times 0 \times \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 \times 0 \\ \times \times \times \\ 0 \times \times \end{pmatrix}. \quad (35)$$

In comparison, the Hermitian M_u and M_d may also have a parallel structure and contain four texture zeros:⁵²

$$M_u = \begin{pmatrix} 0 \times 0 \\ \times \times \times \\ 0 \times \times \end{pmatrix}, \quad M_d = \begin{pmatrix} 0 \times 0 \\ \times \times \times \\ 0 \times \times \end{pmatrix}, \quad (36)$$

where only a single zero does not originate from the basis transformation. But it has been found that a finite (1,1) matrix element of M_u or M_d does not significantly affect the main phenomenological consequences of Eq. (36), if its magnitude is naturally small (.e., $\lesssim m_u$ or $\lesssim m_d$).⁵³

The hierarchical structures of four-zero quark mass matrices in Eq. (36) can be approximately illustrated as follows.⁵⁴

$$\begin{aligned} M_u &\sim m_t \begin{pmatrix} 0 & \vartheta_u^3 & 0 \\ \vartheta_u^3 & \epsilon_u^2 & \epsilon_u \\ 0 & \epsilon_u & 1 \end{pmatrix}, \\ M_d &\sim m_b \begin{pmatrix} 0 & \vartheta_d^3 & 0 \\ \vartheta_d^3 & \epsilon_d^2 & \epsilon_d \\ 0 & \epsilon_d & 1 \end{pmatrix}, \end{aligned} \quad (37)$$

where ϑ_u and ϑ_d essentially correspond to the definitions in Eq. (14), and they are related to ϵ_u and ϵ_d in the following way:

$$\begin{aligned} \vartheta_u^2 &\sim \epsilon_u^6 \sim \frac{m_u}{m_c} \sim 2.2 \times 10^{-3}, \\ \vartheta_d^2 &\sim \epsilon_d^6 \sim \frac{m_d}{m_s} \sim 4.9 \times 10^{-2}. \end{aligned} \quad (38)$$

If the phase difference between M_u and M_d is 90° , then it will be straightforward to obtain $|V_{us}| \simeq |V_{cd}| \sim \sqrt{\vartheta_u^2 + \vartheta_d^2}$ and $|V_{cb}| \simeq |V_{ts}| \sim |\epsilon_u - \epsilon_d|$. While $|V_{td}/V_{ts}| \sim \vartheta_d$ is apparently expected, $|V_{ub}/V_{cb}| \sim \vartheta_u$ must get modified due to the non-negligible contribution of $\mathcal{O}(\vartheta_d^2) \sim \mathcal{O}(\vartheta_u)$ from the down-quark sector. Note that ϵ_u and ϵ_d are not very small,⁵⁴ and their partial cancellation results in a small $|V_{cb}|$ or $|V_{ts}|$.

It is worth pointing out that one may also relax the Hermiticity of quark mass matrices with a number of texture zeros, such that they can fit current experimental data very well.⁵⁵ On the other hand, some texture zeros of quark mass matrices are not preserved to all orders or at any energy scales in a given flavor model. If the model is built at a superhigh-energy scale, where a proper favor symmetry can be used to constrain the structures of quark mass matrices, one has to take account of the renormalization-group running effects in order to compare its phenomenological results with the experimental data at the electroweak scale.³⁸

7. On the Strong CP Problem

So far we have discussed weak CP violation based on the CKM matrix V in the SM. Now let us make a brief comment on the strong CP problem,

because it is closely related to the overall phase of quark mass matrices and may naturally disappear if one of the six quark masses vanishes. It is well known that there exists a P- and T-violating term \mathcal{L}_θ , which originates from the instanton solution to the $U(1)_A$ problem,⁵⁶ in the Lagrangian of QCD for strong interactions of quarks and gluons.⁵⁷ This CP-violating term can be compared with the mass term of six quarks, \mathcal{L}_m , as follows:

$$\mathcal{L}_\theta = \theta \frac{\alpha_s}{8\pi} G_{\mu\nu}^a \tilde{G}^{a\mu\nu} ,$$

$$\mathcal{L}_m = \overline{(u \ c \ t \ d \ s \ b)}_L \mathcal{M} \begin{pmatrix} u \\ c \\ t \\ d \\ s \\ b \end{pmatrix}_R + \text{h.c.} , \quad (39)$$

where θ is a free dimensionless parameter characterizing the presence of CP violation, α_s is the strong fine-structure constant, $G_{\mu\nu}^a$ (for $a = 1, 2, \dots, 8$) represent the $SU(3)_c$ gauge fields, $\tilde{G}^{a\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta} G_{\mu\nu}^a / 2$, and \mathcal{M} stands for the overall 6×6 quark mass matrix. The chiral transformation of the quark fields $q \rightarrow \exp(i\phi_q \gamma_5) q$ (for $q = u, c, t$ and d, s, b) leads to the changes

$$\theta \longrightarrow \theta - 2 \sum_q \phi_q ,$$

$$\arg(\det \mathcal{M}) \longrightarrow \arg(\det \mathcal{M}) + 2 \sum_q \phi_q , \quad (40)$$

where the change of θ follows from the chiral anomaly⁵⁸ in the chiral currents

$$\partial_\mu (\bar{q} \gamma^\mu \gamma_5 q) = 2im_q \bar{q} \gamma_5 q + \frac{\alpha_s}{4\pi} G_{\mu\nu}^a \tilde{G}^{a\mu\nu} . \quad (41)$$

Then the effective CP-violating term in QCD, which is invariant under the above chiral transformation, turns out to be

$$\mathcal{L}_{\bar{\theta}} = \bar{\theta} \frac{\alpha_s}{8\pi} G_{\mu\nu}^a \tilde{G}^{a\mu\nu} , \quad (42)$$

in which $\bar{\theta} = \theta + \arg(\det \mathcal{M})$ is a sum of both the QCD contribution and the electroweak contribution.⁵⁹ The latter depends on the phase structure of the quark mass matrix \mathcal{M} . Because of

$$|\det \mathcal{M}| = m_u m_c m_t m_d m_s m_b , \quad (43)$$

the determinant of \mathcal{M} becomes vanishing in the $m_u \rightarrow 0$ (or $m_d \rightarrow 0$) limit. In this case the phase of $\det \mathcal{M}$ is arbitrary, and thus it can be arranged to cancel out θ such that $\bar{\theta} \rightarrow 0$. Namely, QCD would be a CP-conserving

theory if one of the six quarks were massless. But current experimental data have definitely ruled out the possibility of $m_u = 0$ or $m_d = 0$. Moreover, the experimental upper limit on the neutron electric dipole moment yields $\bar{\theta} < 10^{-10}$.⁶⁰ The strong CP problem is therefore a theoretical problem of how to explain why $\bar{\theta}$ appears but so small.⁶¹

A comparison between weak and strong CP-violating effects might make sense, but it is difficult to find out a proper measure for either of them. The issue involves the reference scale and flavor parameters which may directly or indirectly determine the strength of CP violation. To illustrate,^{47 f}

$$\begin{aligned} \text{CP}_{\text{weak}} &\sim \frac{1}{\Lambda_{\text{EW}}^6} (m_u - m_c)(m_c - m_t)(m_t - m_u)(m_d - m_s) \\ &\quad \times (m_s - m_b)(m_b - m_d) \mathcal{J}_q \sim 10^{-13}, \\ \text{CP}_{\text{strong}} &\sim \frac{1}{\Lambda_{\text{QCD}}^6} m_u m_c m_t m_d m_s m_b \sin \bar{\theta} \sim 10^4 \sin \bar{\theta} < 10^{-6}, \end{aligned} \quad (44)$$

where $\Lambda_{\text{EW}} \sim 10^2$ GeV, $\Lambda_{\text{QCD}} \sim 0.2$ GeV, and the sine function of $\bar{\theta}$ has been adopted to take account of the periodicity in its values. So the effect of weak CP violation would vanish if the masses of any two quarks in the same (up or down) sector were equal^g, and the effect of strong CP violation would vanish if $m_u \rightarrow 0$ or $\sin \bar{\theta} \rightarrow 0$ held. The remarkable suppression of CP violation in the SM implies that an interpretation of the observed matter-antimatter asymmetry of the Universe²³ requests for a new source of CP violation beyond the SM, such as leptonic CP violation in the decays of heavy Majorana neutrino based on the seesaw and leptogenesis mechanisms.⁶⁵

8. Concluding Remarks

Let us make some concluding remarks with the help of the Fritzsche-Xing “pizza” plot as shown in Fig. 4. It offers a summary of 28 free parameters associated with the SM itself and neutrino masses, lepton flavor mixing angles and CP-violating phases. Here our focus is on the five parameters of strong and weak CP violation. In the quark sector, the strong CP-violating phase $\bar{\theta}$ remains unknown, but the weak CP-violating phase δ_q has been determined to a good degree of accuracy. In the lepton sector, however, none of the CP-violating phases has been measured. While the Dirac CP-violating phase δ_ℓ can be determined in the future long-baseline neutrino oscillation

^fWe admit that running the heavy quark masses m_c , m_b and m_t down to the QCD scale might not make sense.⁶² One may only consider the masses of up and down quarks⁶³ and then propose $\text{CP}_{\text{strong}} \sim m_u m_d \sin \bar{\theta} / \Lambda_{\text{QCD}}^2$ as an alternative measure of strong CP violation.

^gIn this special case one of the three mixing angles of V must vanish, leading to $\mathcal{J}_q = 0$ too.⁶⁴

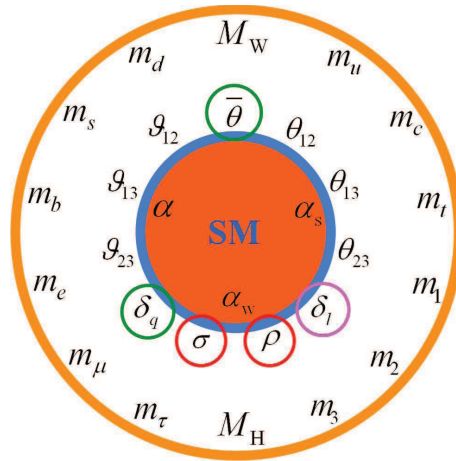


Fig. 4. The Fritzsche-Xing “pizza” plot of 28 parameters associated with the SM itself and neutrino masses, lepton flavor mixing angles and CP-violating phases.

experiments, how to probe or constrain the Majorana CP-violating phases ρ and σ is still an open question.

Perhaps some of the flavor puzzles cannot be resolved unless we finally find out the fundamental flavor theory. But the latter cannot be achieved without a lot of phenomenological and experimental attempts. As Leonardo da Vinci emphasized, “Although nature commences with reason and ends in experience, it is necessary for us to do the opposite. That is, to commence with experience and from this to proceed to investigate the reason.”

Of course, we have learnt a lot about flavor physics from the quark sector, and are learning much more in the lepton sector. We find that the flavors of that big pizza in Fig. 4 are very appealing to us.

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