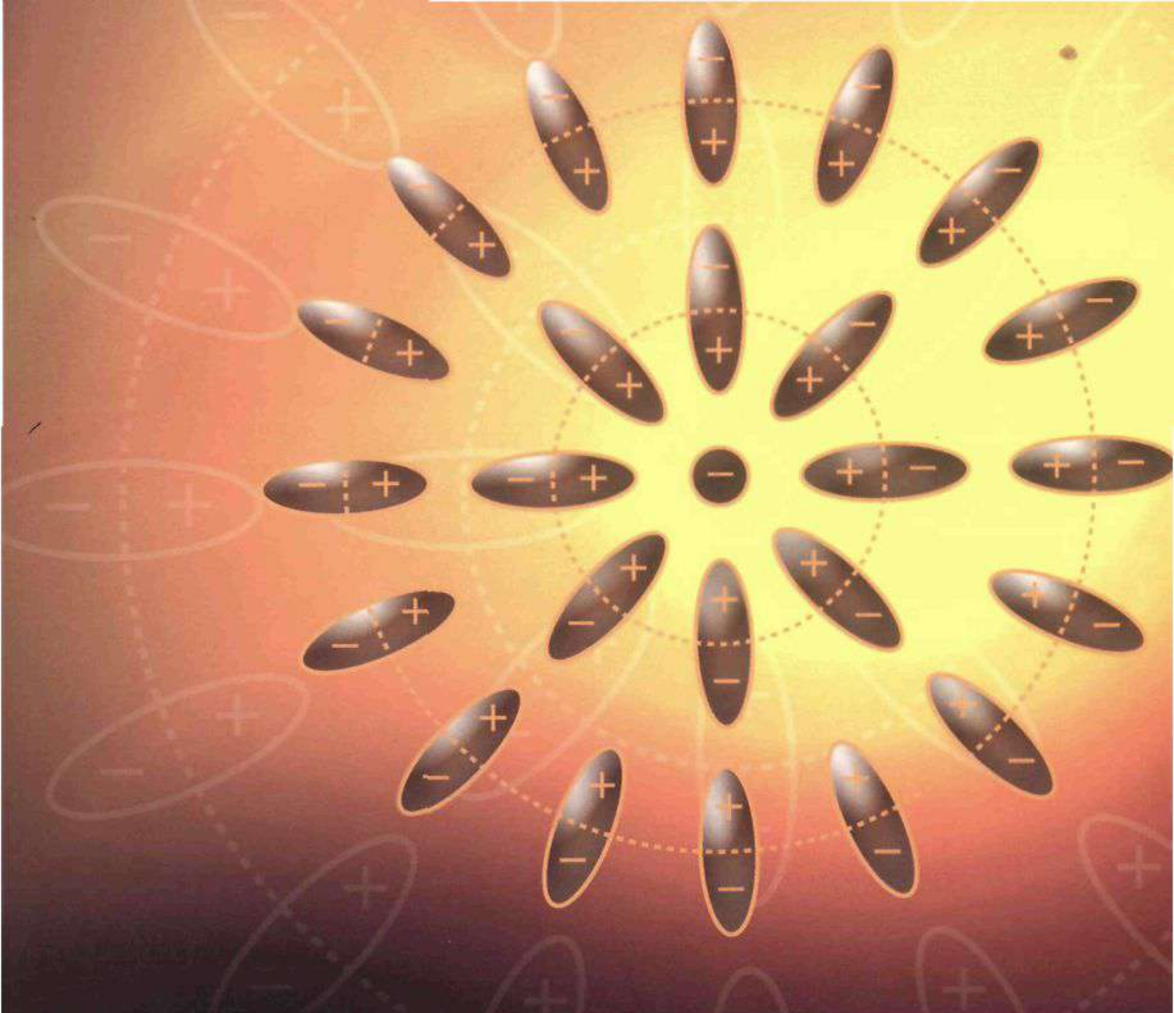

The Physics of the Standard Model and Beyond

T. Morii • C. S. Lim • S. N. Mukherjee

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To our wives, Harue, Yoko and Reba

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PREFACE

Great advances in particle physics have been attained in the last several decades. In the development of the new generation of accelerators and detectors, the quantity and quality of the data on various interaction processes have been vastly increased and our knowledge of particle physics became surprisingly fruitful. Moreover, many interesting and new ideas were proposed and developed successfully in quantum field theories for describing strong, electromagnetic and weak interactions. Those new ideas were concentrated into a simple and beautiful theory known as the standard model in the late 1960's which was formulated in the framework of non-Abelian gauge theory and was extremely successful in describing a wide range of existing phenomena of elementary particles. However, it is also believed that the model is not the ultimate theory; it has many arbitrary parameters which cannot be predicted by the theory and fundamental problems, such as the hierarchy problem, which should be solved in some physics beyond the standard model, "New Physics". Thus it is a general consensus that to search for the evidences for physics beyond the standard model is the most urgent issue in the particle physics of the 21st century.

The aim of the present textbook is to give a unified description of the structures and interactions of elementary particles, by discussing the underlying theories, namely the standard model of elementary particles and physics beyond the standard model. Especially, concerning the description of the physics beyond the standard model, we will select the hot topics of our current interest, including the issues inspired by various experiments as much as possible.

This book has emerged out of the introductory lectures we gave in the graduate course or series of lectures given in various places over several years. Therefore, this book is written primarily for graduate students and can be used by those who wish to major in theoretical and experimental high energy physics. Throughout this textbook, it is assumed that the readers have already gone through a basic course in quantum field theory and particle physics. We hope that this book is also useful for the researchers in the field of particle and nuclear physics.

There are two main features of this textbook. First, we have aimed at a concise description but at the same time we have paid attention so that the basic concepts are clearly mentioned. For such a purpose each section is guided by pedagogical arguments. Second, we also attempt to provide excitement in particle physics, focusing on the important experimental observations (old and new) and a variety of nice ideas for their interpretation.

The organization of the book is as follows. In Chapter 1, we give a brief introduction of the standard model for the elementary particles to describe their structures and interactions. This is followed by Chapter 2 with a description of Fermi theory of weak interactions and its limitation. Preparation for the understanding of electroweak gauge theory is done in Chapter 3, which covers symmetry, symmetry breaking and the gauge theory. Glashow–Weinberg–Salam (GWS) theory of electroweak interactions is covered in Chapter 4. In Chapter 5, quantum chromodynamics (QCD), i.e. the theory of strong interactions, is discussed briefly. After these chapters, we will come to the description of physics beyond the standard model. In Chapter 6, we discuss a current hot topic, neutrino physics, i.e. neutrino masses and neutrino ocsillations. Then, in Chapter 7, we deal with supersymmetric theories as the typical example of physics beyond the standard model. We further attempt to discuss other representative topics in the physics beyond the standard model; precision tests of electroweak radiative corrections and new physics search through these analyses in Chapter 8 and flavor physics and CP violation, which are also sensitive to the presence of new physics, in Chapter 9. Each chapter is accompanied by a few problems, hoping that solving them will be of some help in the understanding of the main text. The topics we chose depend on our preference. However, we believe that those topics must be of general interest and instructive not only for students who are going to enter the field of theoretical and experimental particle physics but also even for researchers in high energy physics. Finally, appendices for some basics are given for the reader's convenience.

During this project, we have learned a lot from many excellent books and reviews; some of them (Bjorken and Drell, 1964; Abers and Lee, 1973; Jauneau, 1977; Quigg, 1983; Cheng and Li, 1984; Halzen and Martin, 1984; Rudaz, 1986; Aitchison and Hey, 1989; Wess and Bagger, 1992; Nagashima, 1999) were good guides in preparing this book.

Unfortunately, because of limited space and time, we could not help neglecting some of the important topics, such as the full contents of QCD, discussion on the grand unified theories (GUT), detailed phenomenological analyses of supersymmetric theories, and so on. Fortunately, concerning these topics, there already exist many excellent textbooks or reviews and we refer the reader to some of them: for example, for comprehensive discussion of QCD, the textbooks by Ynduráin (Ynduráin, 1992) and by Muta (Muta, 1998), for discussion of GUT, the book by Ross (Ross, 1985) and a review by Langacker (Langacker, 1982), for more details on supersymmtry including its phenomenology, the books by Weinberg (Weinberg, 2000) and by Ross (Ross, 1985), and so on. Furthermore, in this textbook the references are far from complete but rather limited only to the ones directly related to the discussions extended in each chapter. We apologize to all authors whose papers we have neglected in spite of their great contributions to the fields.

In carrying out this project, we are deeply indebted to many people, including our teachers, collaborators, students and colleagues who taught and encouraged us a lot. One of the authors (S. N. M.) is grateful to Professor Amitava Roy Chaudhuri for useful comments and to Dr. Sudip Sanyal for research collaboration. The financial assistance obtained from the Department of Science and Technology, New Delhi, India, is also gratefully acknowledged. It took a long time to complete this project; we started at the end of 1993 and finally came to an end in the fall of 2003 after a long struggle. The project was interrupted many times by unexpected difficulties which we came upon in the course of this project, such as the terrible earthquake in Kobe in 1995, the passing away of the mother of one of us (T. M.) in 2002, and so on. In spite of the terrible delay of our work, H. T. Leong, editor of this book, has encouraged us constantly with a warm heart and it is our great pleasure to sincerely thank him for his patient and continuous support for publication of this book. It is also our great pleasure to acknowledge the aid of our graduate students, S. Oyama, K. Mawatari, K. Sudoh, K. Hasegawa and T. Nagasawa, for their self-sacrificing help in drawing and arranging figures and for useful comments. Finally, we are greatly indebted to our families for their warm support and encouragement throughout this long time.

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Kobe, Japan and Vanarasi, India
November 2003

Corrections to this book

Misprints and corrections to this book will be presented on the website at http://www.phys.sci.kobe-u.ac.jp/~lim/bookstb.html. We should be grateful if the reader who would find additional errors or have other comments could kindly send them to lim@phys.sci.kobe-u.ac.jp.

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CONTENTS

Chapter 1

INTRODUCTION

*"From the earliest times, man's dream has been to comprehend the complexity of nature in terms of as few unifying concepts as possible"**

Stars which are twinkling in a night sky have brought us various dreams from ancient times. People have been attracted in their mysterious appearance and have looked for the physical law of the cosmos; how was the cosmos born?, what are the most fundamental building blocks of matter constructing the cosmos?, how do those fundamental particles interact with each other or one another? and so on. There have been many questions to be answered. Particle physics is the most powerful tool for investigating these fundamental questions.

In this introductory chapter, we make a simple explanation on fundamental particles in the standard model and on the characteristic properties of their interactions.

1.1 Elementary particles in the standard model

Elementary particles are the most fundamental building blocks of matter. Entity of elementary particles has been changed in the long course of development of physics. By the end of the last century, people have found that matter is made of molecules and/or atoms, atoms are made of nuclei and electrons, nuclei are made of nucleons, i.e. protons and neutrons, and finally nucleons are made of quarks. Elementary particles of our own time are quarks (together with leptons such as electron). In the development of big accelerators in the last 50 years, many (more than 300) particles called hadrons were discovered in addition to protons, neutrons and pions. There are two kinds of hadrons; baryons (proton, neutron etc.) with the baryon number $B = 1$ and mesons (pions, kaons, etc.) with $B = 0$.

The quark model proposed by Gell-Mann and independently by Zweig in 1964

*Address by Abdus Salam at UNESCO Celebration of the Centenary of Einstein's birth, Paris May 1979.

(Gell-Mann, 1964; Zweig, 1964) classifies all existing hadrons surprisingly well based on the internal symmetry of $SU(3)$ for hadrons composed of relatively light three quarks, and successfully explains static properties of those particles (see Appendix E). In the quark model, baryons are composed of three quarks (qqq) as $p = (uud)$, $n = (udd)$, $\Lambda = (uds)$, etc. and mesons are composed of a quark q and an anti-quark $\bar{q}$ as $\pi^+ = (u\bar{d})$, $\pi^- = (\bar{u}d)$, $K^+ = (u\bar{s})$, $K^- = (s\bar{u})$, etc. Nowadays six different quarks u, d, s, c, b, t are known to exist and thus, it is said that quarks possess 6 degrees of freedom called "flavor". A quark flavor can change into another quark flavor through weak interactions mediated by charged weak bosons $W^\pm$, which are predicted to appear in the electroweak standard model as discussed later in Chapter 4. In addition to flavor, quarks have another degree of freedom called "color" as briefly discussed in Chapter 5. The interaction between quarks due to the color "charge", which is nothing but the strong interaction, is mediated by gluons and is described by quantum chromodynamics(QCD). QCD is the gauge theory with color $SU(3)$ symmetry. While the flavor symmetry is broken by the difference of quark masses, largely for heavy quarks, the color symmetry is an exact symmetry.

Another kind of elementary particles called leptons (electron e, muon μ, tau τ and their corresponding neutrinos ν_e, ν_μ, ν_τ) exist in Nature. Leptons are free from the strong interaction and have no color degrees of freedom, i.e. leptons are colorless. Among them, neutrinos possess only weak interactions, while e, μ and τ have the both of weak and electromagnetic interactions. Productions and decays of leptons are successfully described by the electroweak standard model, i.e. the $SU(2)_L \times U(1)_Y$ gauge theory of electroweak interactions, discussed in chapter 4.

The elementary particles in the standard model are as follows;

$$\text{Quarks} \quad \begin{pmatrix} u \\ d \end{pmatrix}, \quad \begin{pmatrix} c \\ s \end{pmatrix}, \quad \begin{pmatrix} t \\ b \end{pmatrix}, \tag{1.1}$$

$$\text{Leptons} \quad \begin{pmatrix} \nu_e \\ e \end{pmatrix}, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}, \tag{1.2}$$

$$\text{Gauge bosons} \quad \begin{cases} \text{photon} \ \ \gamma, \\ \text{weak (gauge) bosons} \ \ W^\pm, Z^0, \\ \text{gluons} \ \ g, \end{cases} \tag{1.3}$$

$$\text{Higgs bosons} \ \ H. \tag{1.4}$$

Quarks and leptons are fundamental building blocks of matter. All of them are fermions and have spin $\frac{1}{2}$, whose quantum numbers are summarized in Table 1.1 and Table 1.2. It is interesting to note that both quarks and leptons are paired into three doublets, the members of each doublet participating in the charged current weak interaction processes together. The repetition of the doublets is stated as there are three generations of quarks and leptons. The corresponding particles in different

	Q	I_3	S	C	B	T	mass
u	$+\frac{2}{3}$	$+\frac{1}{2}$	0	0	0	0	$1.5 \sim 5$ MeV
d	$-\frac{1}{3}$	$-\frac{1}{2}$	0	0	0	0	$3 \sim 9$ MeV
s	$-\frac{1}{3}$	0	-1	0	0	0	$60 \sim 170$ MeV
c	$+\frac{2}{3}$	0	0	$+1$	0	0	$1.47 \sim 1.83$ GeV
b	$-\frac{1}{3}$	0	0	0	-1	0	$4.6 \sim 5.1$ GeV
t	$+\frac{2}{3}$	0	0	0	0	$+1$	$174.3 \pm 3.2 \pm 4.0$ GeV

Table 1.1 Quarks (Q: electric charge, I_3: 3rd component of isospin, S: strangeness, C: charmness, B: bottomness, T: topness. These quantum numbers change their signs for anti-quarks.)

	Q	L_e	L_μ	L_τ	mass
e	-1	$+1$	0	0	$\cong 0.511$ MeV
ν_e	0	$+1$	0	0	< 3 eV
μ	-1	0	$+1$	0	$\cong 105.66$ MeV
ν_μ	0	0	$+1$	0	< 0.19 MeV
τ	-1	0	0	$+1$	$\cong 1777.0$ MeV
ν_τ	0	0	0	$+1$	< 18.2 MeV

Table 1.2 Leptons (Q: electric charge, L_e: electron number, L_μ: muon number, L_τ: tau number. These quantum numbers change their signs for anti-leptons.)

generations, for instance u, c, t have exactly the same quantum numbers. The only property to distinguish different generations is the difference of the masses of quarks and leptons, depending on the generation. The electric charges of upper and lower components of quark doublets are $+\frac{2}{3}$ and $-\frac{1}{3}$, respectively, while those of lepton doublets are 0 and -1, respectively. At the first sight, such charge "quantization" seems to be miraculous. The quantization, however, is known to be well-suited to the anomaly-free condition for the standard model as explained in Chapter 4.

Gauge bosons having spin 1 are mediators of interactions between quarks or leptons. Interaction strength depends on which gauge bosons propagate between quarks or leptons. Electromagnetic, weak and strong interactions are mediated by photons γ, weak bosons $W^\pm$, Z^0 and gluons g, respectively. A photon does not couple to itself, while gluons and weak bosons have self-couplings, i.e. can couple to themselves. This is due to the fact that the electromagnetic interaction is described by the Abelian gauge theory, while the strong and weak interactions are by the non-Abelian gauge theories. Among those gauge bosons shown in (1.3), a photon γ and gluons g are massless and hence their interaction ranges are infinite, though actually the argument on the strong interaction does not go so straightly because of the non-Abelian nature of color interactions and quarks are confined inside hadrons with the range $\simeq \mathcal{O}(10^{-15}\text{m})$. On the other hand, the weak bosons $W^\pm$, Z^0 are massive and their interactions are very short-range, such as $\frac{1}{m_{W,Z}} \simeq \mathcal{O}(10^{-18}\mathrm{m})$.

The Higgs boson with spin 0 is introduced for the Higgs mechanism to work, which is operative in the theories with spontaneous symmetry breaking of local gauge symmetries as will be discussed in Chapter 3. In the Higgs mechanism a larger symmetry is spontaneously broken into a smaller symmetry through the vacuum expectation value of the Higgs field and accordingly (a part of) gauge bosons become massive. The masses of not only gauge bosons but also all quarks and leptons are originated from the spontaneous symmetry breaking in the standard model, though neutrino masses are a little bit controversial. The real understanding of the mechanism of the spontaneous breakdown and the Higgs mechanism is still extremely challenging problem to be solved in field theories. Obviously, the discovery of the Higgs boson in experiment will provide a crucial hint to the problem. However, so far there is no evidence of the production of the predicted Higgs boson in high energy reactions. To uncover the Higgs boson is the most important issue to finally establish the standard model or even to search for the physics beyond the standard model, "New Physics". This hopefully can be performed in the early time of this century.

1.2 Interactions among fundamental particles

It is well known that there are 4 characteristic interactions among fundamental particles: 1. Electromagnetic interaction mediated by massless photons ($m_\gamma = 0$) with spin=1, 2. Weak interaction mediated by massive weak bosons ($m_W \cong 80.4$ GeV/c^2, $m_Z \cong 91.2$ GeV/c^2) with spin=1, 3. Strong interaction mediated by massless gluons ($m_g = 0$) with spin=1 and 4. Gravitational interaction mediated by massless gravitons ($m_G = 0$) with spin=2. (see Table 1.3.)

Among these interactions, the gravitational interaction is usually out of game for particle physics because it is extremely weak compared with other interactions and has no meaningful effect on any reactions of those particles (,unless the energies of interacting particles are extremely high). For example, the ratio of the gravitational force to the Coulomb (electromagnetic) force between 2 protons at the distance 10^{-15}m is about 10^{-36}.

The electromagnetic interaction mediated by a photon γ has a long history of investigation and now it is known to be described by quantum electrodynamics (QED) which is the gauge theory having the Abelian $U(1)$ symmetry. QED is beautifully formulated in the framework of quantum field theory and is renormalizable, i.e. various divergences originated from the loop integrals in the higher orders of perturbation theory can be renormalized into physical masses and wave functions of particles. Because of smallness of the coupling constant $\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}$, the perturbation works well for QED.

As will be described in Chapter 2, the theory of weak interactions for weak processes originally formulated by Fermi, was developed in 1950's and excellently

Interaction	coupling strength	mediator	spin
1. Electromagnetic	$\alpha = \frac{e^2}{4\pi} \cong \frac{1}{137}$	photon	1
2. Weak	$G_F \cong 1.16 \times 10^{-5}\ \mathrm{GeV}^{-2}$	weak boson	1
3. Strong	$\alpha_s = \frac{g_s^2}{4\pi} \cong 0.1$	gluon	1
4. Gravitational	$G_N \cong 6.71 \times 10^{-39}\ (\mathrm{GeV}/c)^{-2}$	graviton	2

Table 1.3 Particle interactions

described by the current–current interaction with $V - A$ currents. It works well for low energy processes. Unfortunately, the theory is not renormalizable in spite of its small coupling constant. This is due to the fact that the Fermi coupling G_F has the dimension of $[\mathrm{mass}]^{-2}$. Thus the Fermi interaction should be regarded as the effective model for weak processes working only in the low energy region. In the dedicated study of weak interaction physics in 1960's, many theoretical difficulties in the weak interaction were surmounted. A beautiful renormalizable theory was finally formulated, based on the unified picture of weak and electromagnetic interactions, in the framework of non-Abelian gauge theory with $SU(2)_L \times U(1)_Y$ symmetry (the subscript L means the fields participating in the interaction are left-handed and Y denotes the weak hypercharge), which is now called the electroweak standard model. One of the main themes of this textbook is to describe the structure and physics of this model.

The strong interaction is mediated by gluons which have color charges. Since quarks have also color charges, gluons can couple to quarks. The field theory for the strong interaction is formulated in the non-Abelian gauge theory with $SU(3)_c$ color symmetry and is called quantum chromodynamics(QCD). The coupling constant of QCD has conspicuous behavior for a variation of momentum transfer square Q^2, as briefly described in Chapter 5. The strong coupling constant $\alpha_s(Q^2) = \frac{g_s^2}{4\pi}$ "runs" as Q^2 varies. On one hand, $\alpha_s(Q^2)$ becomes small for large Q^2 region as realized in hard scattering such as deep inelastic scattering, where quarks and gluons behave as free particles, implied by the word "asymptotic-free", and in such regions the perturbation theory works well. On the other hand, for small Q^2 region as realized in the static state of bound quarks inside hadrons, $\alpha_s(Q^2)$ becomes large and in this region the perturbative treatment is not reliable, where quarks are confined inside hadrons (in color singlet states). This is called the "confinement" phase. QCD must be the theory for describing the dynamics of quarks and gluons in all Q^2 regions from "asymptotic-free" to "confinement" phases.

Weak and electromagnetic interactions are formulated by the gauge theory with $SU(2)_L \times U(1)_Y$ symmetry and furthermore, the strong interactions are described by the gauge theory with color $SU(3)_c$ symmetry. Hence, one can naturally expect that all these interactions of elementary particles must be described by the gauge theory with some internal symmetry G, that is, the Lagrangian has to be invariant under the gauge transformations of G. The simplest example is to take the symmetry group G to be a direct product of each symmetry, $G = SU(3)_c \times SU(2)_L \times U(1)_Y$. The resultant theory is called the standard model of particle physics. The important principle in the formulation is that (1)the theory is the gauge theory, (2)it must be renormalizable and anomaly-free and (3)the symmetry breaking must occur spontaneously.

Unfortunately, the standard model has many problems to be solved and many theorists believe that it is not the ultimate theory. To solve these remaining problems and go further, we must look for evidences for the physics beyond the standard model, "New Physics". The discussion of the New Physics is another main theme of this textbook and will be discussed in detail from Chapter 6 to Chapter 9. The issues discussed in these Chapters are selected from the challenging hot topics of our current interest.

Chapter 2

WEAK INTERACTION

The theoretical description of weak interaction processes was first worked out by Fermi in 1933 for the β-decay of nuclei (Fermi, 1933). The following two decades saw the refinement of the explicit form of the current–current interaction first suggested by Fermi. The discovery of parity violation by Wu et al. in 1957 (following the theoretical work of Lee and Yang) led Feynman and Gell-Mann and also Marshak and Sudarshan to the vector minus axial vector $V - A$ structure of the weak current in 1958 (Feynman and Gell-Mann, 1958; Marshak and Sudarshan, 1958). In 1961, Gell-Mann and Neéman independently introduced the $SU(3)$ symmetry of strong interaction (Gell-Mann and Neéman, 1964) and in 1963, Cabibbo (Cabibbo, 1963) used the hypothesis that the weak currents of hadrons have definite $SU(3)$ transformation properties to describe the decay of strange particles in the current–current coupling scheme.

In this Chapter, after touching on the parity violation in weak interaction processes, we describe the development of weak interactions following the above historical chronology. We also discuss some important subjects in the Fermi theory, i.e. the lepton current universality, pion decays and Cabibbo currents. Unfortunately, the Fermi theory is not complete. It has serious fundamental difficulies, i.e. unitarity violation and non-renormalizability. These difficulties will be discussed in the subsequent section. Finally, we will give some discussions on the intermediate weak boson model, which is an alternative model of the Fermi theory, as the preliminary discussion of the Standard Model.

2.1 The Fermi Theory of Weak Interaction

The weak interaction among elementary particles was first discovered in the β-decay of nuclei. Basically it is realized as a decay of neutron n in the nucleus into a proton p, an electron e^- and an anti-neutrino $\bar{\nu}$,

$$n \to p + e^- + \bar{\nu}. \tag{2.1}$$

The life of a free neutron due to the β-decay is about 900sec which is extremely longer than that of other particles induced by strong interaction like $\tau(\rho^0 \to \pi^+\pi^-) \sim 10^{-24}$sec or electromagnetic interaction like $\tau(\pi^0 \to 2\gamma) \sim 10^{-18}$sec. Since the nuclear β-decays share common nature with other weak processes, they are classified into weak interaction processes, though the neurton life-time is much larger than that of typical weak decays like $\tau(\Lambda^0 \to p\pi^-) \sim 10^{-10}$sec.

2.1.1 *Parity Violation in Weak Interaction*

One of the most conspicuous properties of weak interactions is parity violation. Before 1956 when Lee and Yang (Lee and Yang, 1956) proposed an idea of parity violation in particle physics, people had considered that all physical processes were invariant under space inversion and thus parity was a good quantum number for all interactions. After the experimental discovery of parity violation by Wu et al. in 1957 (Wu et al, 1957), much progress was made for determining the weak interaction type which was finally established as $V - A$ interactions. The story of this progress is exciting and will be discussed in the next section. But before coming into this history, here we would like to touch on the story of discovery of parity violation.

It is not in the β-decay of neutron but in the K^+ decay that Lee and Yang suggested first an idea of parity violation. K^+ meson decays into $\pi^+\pi^0$ (Br = 21.1%) and also even into $\pi^+\pi^+\pi^-$ (Br = 5.6%), where Br denotes the branching ratio. Since a π meson is pseudoscalar, its intrinsic parity is -1. The spin of the K meson and π meson is 0 in either case. Therefore, for $\pi^+\pi^0$ decays, the orbital angular momentum of $\pi^+\pi^0$ is 0 because of total angular momentum conservation and thus the parity of the $\pi^+\pi^0$ system becomes $+1$. On the other hand, for the $\pi^+\pi^+\pi^-$ system, the total orbital angular momentum of this system is given by the sum of the orbital angular momentum between two π^+ and the one between the center of mass of two π^+ and the remaining π^-. The sum should be 0 because it must be equal to the spin of the K meson which is 0. This can be realized only when the magnitude of these two orbital angular momenta are equal. Therefore, the parity of this system due to the orbital angular momentum becomes $+1$. However, since the intrinsic parity of π meson is -1, the intrinsic parity of the three pion system is $(-1)^3 = -1$ and hence the parity of $\pi^+\pi^+\pi^-$ becomes -1, after all. Thus, the parity of $\pi^+\pi^0$ is $+1$, while that of $\pi^+\pi^+\pi^-$ is -1.

Then, if parity is conserved in these decays, we must have two independent K mesons, say θ^+ and τ^+; one is of even parity and another is of odd parity. In fact, in those days the decays were understood as $\theta^+ \to \pi^+\pi^0$ and $\tau^+ \to \pi^+\pi^+\pi^-$. It, however, seems to be too accidental that these two particles θ^+ and τ^+ have the same mass, the same life-time, the same spin and so on. This problem was known as the $\tau - \theta$ puzzle in those days. To solve the puzzle, Lee and Yang suggested that the parity is violated in weak decay processes. If this is the case for any weak

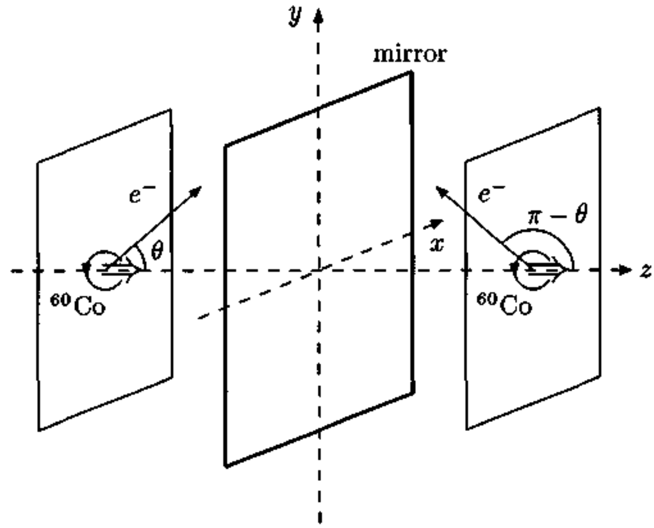


Fig. 2.1

interaction processes, the effect should appear even in the β-decay of nucleus. They proposed to observe the correlation between the spin $\vec{J}$ of ^{60}Co and the momentum $\vec{p}_e$ of elecron produced in the following β-decay,

$$^{60}\mathrm{Co}(J^p = 5^+) \rightarrow^{60} \mathrm{Ni}(4^+) + e^- + \bar{\nu}. \tag{2.2}$$

First, align the spin $\vec{J}$ of ^{60}Co in a $+z$ direction perpendicular to a fixed $x - y$ plane and then set a mirror in parallel to the $x - y$ plane at some point z (Fig. 2.1). The direction of spin $\vec{J}$ of ^{60}Co in the mirror image is the same as that of the real ^{60}Co because of axialvector nature of $\vec{J}$. Then, when an electron is produced in a direction, forming an angle θ relative to $\vec{J}$, the electron in the mirror image is produced in a relative angle $\pi - \theta$ because of vector nature of $\vec{p}_e$ as shown in Fig. 2.1. Therefore, if the process is parity invariant, the probability for finding electrons produced with the angle θ and $\pi - \theta$ must be the same. Wu and her collaborators observed that the probability for finding electrons produced with $\theta = \pi$ was much larger than the one for the case of $\theta = 0$. This was the first discovery of parity violation.

2.1.2 *Road to Current–Current $V - A$ Interaction*

In 1933, Fermi proposed an idea that the light leptons e^- and $\bar{\nu}$ are emitted from the neutron n in the β-decay (Fig. 2.2(a)) like a photon γ emitted from radioactive nuclei, and treated the process based on quantum field theory. In quantum electrodynamics, the radiative process $p \rightarrow p + \gamma$ (Fig.2.2(b)) where p is a proton in the

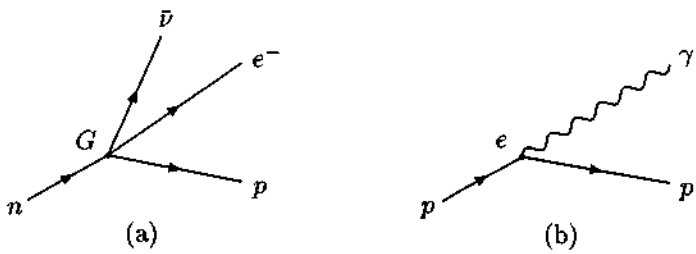


Fig. 2.2

nucleus is described by

$$ej_\mu^{(em)}A^\mu = e\left[\bar{u}_p\gamma_\mu u_p\right]A^\mu, \tag{2.3}$$

where e is the electric charge of proton and u_p and A^μ are the Dirac spinor of proton and the photon field, respectively. The factor $j_\mu^{(em)} = \bar{u}_p\gamma_\mu u_p$ is called an electromagetic current of proton. To describe the β-decay process $n \to p + e^- + \bar{\nu}$, Fermi replaced A^μ and the electromagnetic current $j_\mu^{(em)}$ by a 4-vector weak lepton current $j^\mu_{(\nu\to e)} = \bar{u}_e\gamma^\mu u_\nu$ made of e^- and ν fields and the weak nucleon current $j_\mu^{(n\to p)} = \bar{u}_p\gamma_\mu u_n$ of p and n fields, respectively, as

$$Gj_\mu^{(n\to p)}j^\mu_{(\nu\to e)} = G\left[\bar{u}_p\gamma_\mu u_n\right]\left[\bar{u}_e\gamma^\mu u_\nu\right], \tag{2.4}$$

where G is the weak coupling constant being much smaller than the electromagnetic coupling constant e. (Actually $GE^2 \ll e^2$ for moderate energies E.) That is, the β-decay interaction is described by the product of two weak currents; nucleon current $j_\mu^{(n\to p)}$ and lepton current $j^\mu_{(\nu\to e)}$, in which 4 fermions p, n, e^- and $\bar{\nu}$ couple at the same space-time point. There is no propagator connecting these two currents and this is apparently different from electromagnetic processes where, for instance, proton current and electron current are mediated by a photon with its propagator. In addition to a big difference in magnitude of G and e, it is remarkable that the weak nucleon and lepton currents are charged,* while the electromagnetic currents are neutral.

Taking account of the parity violation in weak interaction, Fermi's original idea for the β-decay $n \to p + e^- + \bar{\nu}$ can be generalized by writing the weak interaction Hamiltonian for this process as (see, for example, Jauneau, 1977)

$$H_w = \sum_i \frac{G_i}{2}\left[\bar{\psi}_p O_i \psi_n\right]\left[\bar{\psi}_e O_i(1 + c_i\gamma_5)\psi_\nu\right] + h.c., \tag{2.5}$$

where c_i can be taken to be ± 1 since the parity violation is maximum and actually

*Long later, in 1973 the weak neutral current was also discovered with neutrino elastic scattering $\bar{\nu} + e^- \to \bar{\nu} + e^-$ and deep inelastic scattering $\nu + N \to \nu + X$ and $\bar{\nu} + N \to \bar{\nu} + X$ as predicted by the Glashow–Weinberg–Salam(GWS) theory.

c_i becomes -1 because the neutrino is left-handed, as discussed later in 2.1.2.3. G_i is the coupling constant for the type $i(= S, V, T, A, P)$ corresponding to the specific Lorentz structure of interactions and O_i is given by

$$\begin{aligned} O_S &= 1, && \text{Scalar}(S) \\ O_V &= \gamma_\mu, && \text{Vector}(V) \\ O_T &= \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), && \text{Tensor}(T) \\ O_A &= \gamma_5\gamma_\mu, && \text{Axialvector}(A) \\ O_P &= \gamma_5. && \text{Pseudoscalar}(P) \end{aligned} \tag{2.6}$$

Assuming the time reflection invariance in the β-decay, we can take G_i to be real. Then, the remaining task is to determine experimentally which type of interaction among (2.6) works in the β-decay.

2.1.2.1 *Helicity and chirality*

Let us start with the plane wave of a fermion with momentum $\vec{p}$, energy E and mass m. The equation of motion of this particle is given by the Dirac equation, in the natural unit, $\hbar = c = 1$,

$$(\vec{\alpha} \cdot \vec{p} + \beta m)\psi = E\psi, \tag{2.7}$$

where $\vec{\alpha}$ and β are 4×4 matrices and related to γ-matrices as $\gamma^0 = \beta$ and $\vec{\gamma} = \beta\vec{\alpha}$. For a massless particle with $m = 0$ as in the case of neutrino†, we have

$$\vec{\alpha} \cdot \vec{p}\psi = p\psi. \tag{2.8}$$

In the γ^0-diagonal representation, $\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ (see (A.25)) and by introducing a 4×4 matrix $\vec{\sigma}' = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$, one can write $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$ as $\vec{\alpha} = \gamma_5\vec{\sigma}' = \vec{\sigma}'\gamma_5$, where $\vec{\sigma}$ is the Pauli matrix. Now using $\gamma_5^2 = 1$, one can obtain the following equations from (2.8)

$$\frac{\vec{\sigma}' \cdot \vec{p}}{p}\psi = \gamma_5\psi, \quad \frac{\vec{\sigma}' \cdot \vec{p}}{p}\gamma_5\psi = \psi, \tag{2.9}$$

†Recently, due to observation of neutrino oscillation, it has been strongly suggested that neutrinos are massive. But in those days neutrinos were considered to be massless and here we treat massless neutrinos. In any case, neutrino mass has to be extremely small even if it is massive and it is a good approximation to consider the massless neutrinos.

Then, by taking addition and subtraction of two equations in (2.9) and introducing the helicity operator defined by $h = (\vec{\sigma} \cdot \vec{p})/2p$, we obtain from (2.9)

$$h\psi_R = \frac{1}{2}\psi_R, \qquad h\psi_L = -\frac{1}{2}\psi_L, \tag{2.10}$$

where $\psi_R = P_R\psi$ with $P_R = (1 + \gamma_5)/2$ and $\psi_L = P_L\psi$ with $P_L = (1 - \gamma_5)/2$ and they are called right- and left-handed fermions, respectively. P_R and P_L are projection operators for right- and left-handed fermions, respectively. (2.10) shows that ψ_R and ψ_L are eigenstates of helicity with eigenvalues $h = +1$ and $h = -1$, respectively[‡]. $h = +1$ means that the particle spin is parallel to the direction of its momentum $\vec{p}$ and $h = -1$ means that the particle spin is anti-parallel to its momentum $\vec{p}$.

γ_5 is called chirality operator and the following eigenvalue equations hold for any fermions with or without masses

$$\gamma_5\psi_R = \psi_R, \qquad \gamma_5\psi_L = -\psi_L. \tag{2.11}$$

Comparing (2.10) with (2.11) we see that helicity h is the same as chirality γ_5 for massless fermions. This is true even in the γ_5-diagonal representaion. (see Appendix A.5.)

Now let us consider a free electron with nonzero mass m. We are interested in the effect of the projection operators $P_{R,L}$ on the Dirac spinor of the electron, which is the solution of (2.7)

$$\psi = N \begin{pmatrix} \chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi \end{pmatrix}, \tag{2.12}$$

where N is the normalization factor and $\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represent that the electron spin is parallel and anti-parallel to the z-axis, respectively. With (2.12), we obtain

$$\psi_R = \frac{N}{2} \begin{pmatrix} \left(1 + \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\chi \\ \left(1 + \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\chi \end{pmatrix}, \qquad \psi_L = \frac{N}{2} \begin{pmatrix} \left(1 - \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\chi \\ \left(-1 + \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right)\chi \end{pmatrix}. \tag{2.13}$$

Now let the electron momentum $\vec{p}$ be along the z-axis, for simplicity. Then, we can calculate the expectation value of helicity $h^{(e)}$ of electron as

$$P^{(e)} = \frac{< \psi_{R(L)}|h^{(e)}|\psi_{R(L)} >}{< \psi_{R(L)}|\psi_{R(L)} >} = \pm\frac{p}{E}, \tag{2.14}$$

[‡]Though according to the definition of helicity operator, the face value of helicity eigenvalue for fermions with spin $\frac{1}{2}$ is $h = \frac{1}{2}$ or $h = -\frac{1}{2}$, the two times value of it is conventinally used and we follow this convention in this textbook.

where the sign $+(-)$ corresponds to $\psi_{R(L)}$. As $\frac{p}{E}$ equals to v (in the natural unit $c=1$), with v being the velocity of electron, $P^{(e)}$ approaches to ± 1 for relativistic electron with $v \simeq 1$. In other word, $P_{R(L)}$ picks up approximately the $h^{(e)} = +1(-1)$ state for a high energy electron(see also Appendix A.5).

2.1.2.2 *Observation of electron helicity in the β-decay*

In 1957, Frauenfelder et al. observed the longitudinal polarization of electron $P^{(e)}$ in (2.14) in the β-decay ${}^{60}\mathrm{Co} \to {}^{60}\mathrm{Ni} + e^- + \bar{\nu}$, which was measured to be $-v$, i.e. chirality of electron was approximately left-handed (Frauenfelder et al, 1957). There are several methods to observe the longitudinal polarization of electron. Many experiments were performed in those days and the conclusion was that electrons have negative helicity (left-handed) while positrons, i.e. anti-particles of electrons, have positive helicity (right-handed).

If electrons are left-handed, we can replace $\psi_e \to \frac{1-\gamma_5}{2}\psi_e$ and also $\bar{\psi}_e \to \bar{\psi}\frac{1+\gamma_5}{2}$ in the Fermi interaction (2.5). Then the lepton current factor $\bar{\psi}_e O_i(1+c_i\gamma_5)\psi_\nu$ becoms

$$\begin{aligned}
\bar{\psi}_e O_i(1+c_i\gamma_5)\psi_\nu &\to \bar{\psi}_e \frac{1+\gamma_5}{2} O_i(1+c_i\gamma_5)\psi_\nu \\
&= \bar{\psi}_e O_i \frac{1\mp\gamma_5}{2}(1+c_i\gamma_5)\psi_\nu \\
&= \bar{\psi}_e O_i(1\mp c_i)\frac{1\mp\gamma_5}{2}\psi_\nu, \qquad (2.15)
\end{aligned}$$

where we used the relation $\gamma_\mu\gamma_5 = -\gamma_5\gamma_\mu$. The sign in the right-hand side of (2.15) shows that $-$ is for $i = V, A$ and $+$ for $i = S, T, P$. This equation is very important in order to determine the type of O_i. If neutrino helicity is measured to be -1, one can say that the interaction type must be V or A or a combination of them, and c_i becomes -1($c_i = +1$ is not allowed). On the contrary, if it is $+1$, the interaction type should be S, T, P or a combination of them, and c_i become $+1$($c_i = -1$ is not allowed). Therefore, the next step is to know neutrino helicity which is necessary to determine the type of weak interactions.

2.1.2.3 *Determination of neutrino helicity*

In 1958, Goldhaber et al. directly measured neutrino helicity in their exquisite experiment (Goldhaber et al., 1958). To measure neutrino helicity, they studied the K-capture decay of ${}^{152}\mathrm{Eu}$ to ${}^{152}\mathrm{Sm}$; ${}^{152}\mathrm{Eu}$ first absorbs an electron in its atom, which is called K-capture since an electron is absorbed from the K-orbit, and decays into an excited state of ${}^{152}\mathrm{Sm}^*$ and a neutrino ν, and then finally ${}^{152}\mathrm{Sm}^*$ comes to its ground state by emitting a photon γ. After all, the total reaction is as follows;

$$ {}^{152}\mathrm{Eu}(J^P = 0^-) + e^- \to {}^{152}\mathrm{Sm}^*(1^-) + \nu \to {}^{152}\mathrm{Sm}(0^+) + \gamma + \nu. \qquad (2.16)$$

In this process, the produced neutrino cannot be detected because it is neutral and has no electromagnetic interaction.

Now, let us consider a photon which runs only in the direction parallel to $^{152}\mathrm{Sm}(0^+)$ in the final state. If we take the direction of running neutrino as $+z$-axis, we see that only two cases of spin alignment are allowed in this process:

$$\begin{array}{lccccccc} & ^{152}\mathrm{Eu}(0^-) & + e^- & \rightarrow & ^{152}\mathrm{Sm}(0^+) & + \gamma & + \nu & \\ \text{(i)} & 0 & -\frac{1}{2} & & 0 & -1 & +\frac{1}{2} \\ \text{(ii)} & 0 & +\frac{1}{2} & & 0 & +1 & -\frac{1}{2} \end{array} \tag{2.17}$$

If $h^{(\nu)} = +1$, i.e. $J_z^{(\nu)} = +\frac{1}{2}$, then $J_z^{(\gamma)} = -1$ for the produced photon because of the angular momentum conservation. It means that the produced photon is right-circularly polarized, since it is running along $-z$ direction. On the contrary, if $h^{(\nu)} = -1$, i.e. $J_z^{(\nu)} = -\frac{1}{2}$, then the produced photon must be left-circularly polarized. The key point of this experiment was that they determined neutrino helicity $h^{(\nu)}$ by measuring the photon circular (left- or right-) polarization. The observed photon was left-circularly polarized, which led to $h^{(\nu)} = -1$. This results in $c_i = -1$ as seen from (2.15) with maximum parity violation. From this result, the lepton current in (2.5) becomes a $V - A$ type, i.e. $\bar{\psi}_e\gamma^\mu(1-\gamma_5)\psi_\nu$.

From the above observation, we can say that the allowed types of the nucleon current for β-decay processes are also $O_i = V$ and A because of Lorentz invariance of the interaction Hamiltonian (2.5). Historically, it has been well known that in the nuclear β-decay there are two types of transitions, i.e. (1)Fermi transition and (2)Gamow–Teller transition, depending on the spin of the lepton pair $e^-\bar{\nu}$ which can be singlet($S = 0$) and triplet($S = 1$) states, respectively. In the nonrelativistic approximation which works for ordinally β-decay processes, one can easily find that Fermi transitions occur for $O_i = S, V$, while Gamow–Teller transitions do for $O_i = T, A$. Moreover, $O_i = P$ contributes to Fermi transitions only in the order of $\frac{v}{c^2}$ and thus can be neglected in the nuclear β-decay. The process discussed here is the Gamow–Teller transition and hence the above result tell us that $O_i = A$ is allowed but T is not allowed for the nucleon current of this process.

2.1.2.4 *Angular correlation between e^+ and ν*

Another interesting experiment for studying the interaction type of β-decays was measurement of angular correlation between e^+ and ν in the decay $^{35}\mathrm{A} \rightarrow ^{35}\mathrm{Cl} + e^+ + \nu$ which was observed by Allen et al. (Hermannsfeldt et al, 1957). It is known that this process is Fermi transition (spin of the lepton pair $e^+\nu$ being 0) and only $O_i = V$ or S contribute to this transition (Kistner at al, 1956). In those days before the experiment by Goldhaber et al. (1958) which established the neutrino helicity to be $h^{(\nu)} = -1$, people were interested in which one of V and S is chosen by Nature.

Let us take the direction of the emitted positron e^+ to be the positive z-axis and consider two possible cases; e^+ and ν are emitted (1)in parallel with $\theta = 0$, where

θ is an angle between momenta of e^+ and ν, and (2)in anti-parallel with $\theta = \pi$. Since the helicity of e^+ is $+1$, the angular momentum of e^+ is $J_z = +\frac{1}{2}$. Then, if the helicity of neutrino is negative, i.e. $h^{(\nu)} = -1$, we can expect, from (2.15) and the angular momentum conservation, that the case (1) is possible only for $O_i = V$ but is not allowed for S and the case (2) does not occur for the Fermi transition with $O_i = V$. That is to say, it is forbidden for e^+ and ν to be emitted in parallel if $O_i = S$. On the contrary, if $h^{(\nu)} = +1$, the case (2) can occur and this time $O_i = S$ is possible and V is not allowed from (2.15). Allen et al. observed that the case (1) was realized.

From those many experimental observations, it was concluded that the weak interaction of the β-decay is given by a combination of V and A as follows;

$$\begin{aligned} H_w &= \frac{G_V}{2}\left[\bar{\psi}_p\gamma_\mu\psi_n\right]\left[\bar{\psi}_e\gamma^\mu(1-\gamma_5)\psi_\nu\right] \\ &+ \frac{G_A}{2}\left[\bar{\psi}_p\gamma_5\gamma_\mu\psi_n\right]\left[\bar{\psi}_e\gamma^\mu(1-\gamma_5)\psi_\nu\right] + h.c. \end{aligned} \tag{2.18}$$

2.1.2.5 $V - A$ *interaction*

Experimentally, G_V was determined from the decay $^{16}\mathrm{O}(0^+) \rightarrow ^{14}\mathrm{N}(0^+) + e^+ + \nu$. In this process, the spin of initial and final nucleons does not change and the spin sum of e^+ and ν is 0. Thus, the process is Fermi transition and hence only G_V in (2.18) can contribute to this decay and then from this decay width one can obtain the value of G_V; when we define $G_\beta = G_V/\sqrt{2}$, we obtained

$$G_\beta = 1.147 \times 10^{-5}\mathrm{GeV}^{-2} \approx 10^{-5}/m_p^2, \tag{2.19}$$

where m_p is the mass of proton being the order of 1 GeV. Using this value, G_A can be obtained, in principle, from the β-decay of neutron, where the Fermi transition and the Gamow-Teller transition coexist. There was a long history of this measurement and now we have $g_A = |G_A/G_V| \simeq 1.26$. Unfortunately, the relative sign of G_V and G_A, cannot be determined from unpolarized experiments. To determine it, one must study the polarized experiment.

Now, let us consider the β-decay of neutron whose spin is polarized to the $+z$-axis and let electron and anti-neutrino momenta be given by $\vec{p}_e$ and $\vec{p}_{\bar{\nu}}$, respectively. Since the mass difference between neutron and proton is tiny, the produced proton's momentum is very small and thus, it is a good approximation to treat the nucleon current nonrelativistically. First consider the first term of (2.18), i.e. the Fermi transition term. Using the nucleon wave function which is of the same form as (2.12) with almost equal mass $m_p \approx m_n \approx m$ for proton and neutron, we see that $\bar{\psi}_p\gamma_0\psi_n$ is of $\mathcal{O}(1)$, while $\bar{\psi}_p\vec{\gamma}\psi_n$ is of $\mathcal{O}(\frac{v}{c^2})$ in the nonrelativistic approximation. Thus, we can neglect the contribution of $\bar{\psi}_p\vec{\gamma}\psi_n$ and the remaining term is only $\bar{\psi}_p\gamma_0\psi_n$ which can be written as $\chi_p^\dagger\chi_n$ in the nonrelativistic approximation using the Pauli spinor $\chi_{p(n)}$ of proton(neutron). It has no spin flip operator $\vec{\sigma}$ and therefore, in this case only two patterns of the spin and momentum arrangement for particles

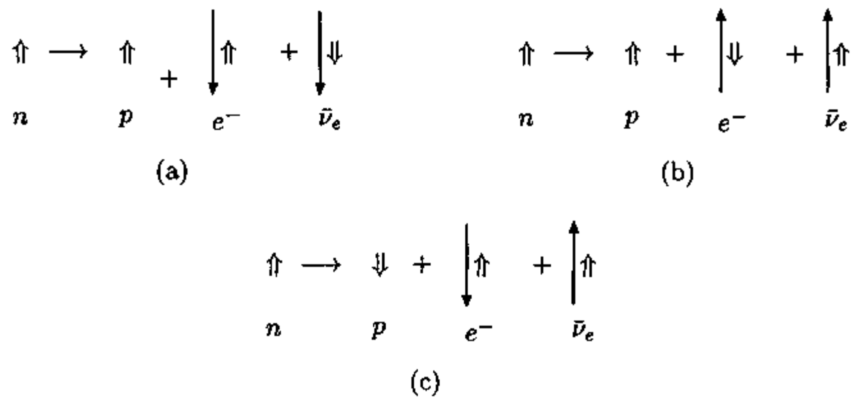


Fig. 2.3 Spin arrangement in the neutron β-decay. Long arrows show the momenta of e^- and $\bar{\nu}_e$

participating in this β-decay can be allowed as shown in Fig. 2.3(a) and (b). Next, let us consider the second term of (2.18), i.e. the Gamow-Teller transition term. This time the same wave function of nucleons leads to that $\bar{\psi}_p\vec{\gamma}\gamma_5\psi_n$ is of $\mathcal{O}(1)$, while $\bar{\psi}_p\gamma_0\gamma_5\psi_n$ is of $\mathcal{O}(\frac{v}{c^2})$. Thus, we can neglect $\bar{\psi}_p\gamma_0\gamma_5\psi_n$. Only $\bar{\psi}_p\vec{\gamma}\gamma_5\psi_n$ contributes to this transition. $\bar{\psi}_p\vec{\gamma}\gamma_5\psi_n$ results in $\chi_p^\dagger\vec{\sigma}\chi_n$ in the nonrelativistic approximation. For σ_3 term in $\chi_p^\dagger\vec{\sigma}\chi_n$, the spin and momentum arrangement of particles are same as shown in Fig. 2.3(a) and (b). However, σ_1 and σ_2 terms contribute to spin flip and thus we have the spin and momentum arrangement as shown in Fig. 2.3(c).

After all, in the nonrelativistic approximation we have the following interaction Hamiltonian for the β-decay of neutron.

$$\begin{aligned} H_w &= G_V\left[\chi_p^\dagger\chi_n\right]\left[\eta_e^\dagger\eta_\nu\right] \\ &+ G_A\left[\chi_p^\dagger\vec{\sigma}\chi_n\right]\left[\eta_e^\dagger\vec{\sigma}\eta_\nu\right] + h.c., \end{aligned} \tag{2.20}$$

where $\frac{1-\gamma_5}{2}\psi_\nu$ in (2.18) is replaced by η_ν being the two-component neutrino wave function with helicity $h^{(\nu)} = -1$ and η_e is also the two-component Weyl spinor of electron with helicity $h^{(e)} = -1$ (see (A.53) for the notation of two–component Weyl spinors). Now, let us calculate the matrix element for the process presented in Fig. 2.3(c) where the neutron spin flips. The spin flip is originated from the following terms in (2.20),

$$\begin{aligned} &\left[\chi_p^\dagger\sigma_1\chi_n\right]\left[\eta_e^\dagger\sigma_1\eta_\nu\right] + \left[\chi_p^\dagger\sigma_2\chi_n\right]\left[\eta_e^\dagger\sigma_2\eta_\nu\right] \\ &= 2\left[\chi_p^\dagger\sigma_+\chi_n\right]\left[\eta_e^\dagger\sigma_-\eta_\nu\right] + 2\left[\chi_p^\dagger\sigma_-\chi_n\right]\left[\eta_e^\dagger\sigma_+\eta_\nu\right], \end{aligned} \tag{2.21}$$

where $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$ and actually only the 2nd term of (2.21) can contributes to the spin arrangement presented in Fig. 2.3(c). We can parametrize this matrix element as

$$M^{(c)} = 2G_AF, \tag{2.22}$$

where F is a factor containing all dynamics of this decay process except for the coupling constant G_A.

Next, let us consider the case of Fig. 2.3(a). This time, the 1st term of (2.20) and the σ_3 term in the 2nd term of (2.20) contribute. In Fig. 2.3(a), the electron spin being in parallel to the neutron spin is along the $+z$-axis. Hence σ_3 works as $+1$ in the matrix element and we can write the matrix element for this process as

$$M^{(a)} = (G_V + G_A)\,F. \tag{2.23}$$

For the case of Fig. 2.3(b), since the electron spin is along the $-z$-axis, we obtain the following matrix element,

$$M^{(b)} = (G_V - G_A)\,F. \tag{2.24}$$

Then, we can calculate the probability for an electron to be emitted in parallel to the neutron spin as shown in Fig. 2.3(b), as

$$P(\vec{\sigma}\uparrow\vec{p}_e\uparrow) = |M^{(b)}|^2 = |G_V - G_A|^2|F|^2. \tag{2.25}$$

To the contrary, the probability for an electron being emitted in anti-parallel to the neutron spin, which is given by the sum of Fig. 2.3(a) and (c), can be calculated as

$$P(\vec{\sigma}\uparrow\vec{p}_e\downarrow) = |M^{(a)}|^2 + |M^{(c)}|^2 = |G_V + G_A|^2|F|^2 + 4|G_A|^2|F|^2. \tag{2.26}$$

If $G_V \approx G_A$, then $P(\vec{\sigma}\uparrow\vec{p}_e\uparrow) \approx 0$, which means that an electron cannot be emitted in parallel to the neutron spin. If $G_V \approx -G_A$, then $P(\vec{\sigma}\uparrow\vec{p}_e\uparrow) \approx P(\vec{\sigma}\uparrow\vec{p}_e\downarrow)$, which means that the number of electrons being emitted in parallel and anti-parallel to the neutron spin is almost same. The latter case was observed in experiment and then it was finally established that the weak interaction Hamiltonian for the β-decay was described by the $V-A$ current–current interactions as summarized as follow;

$$H_w = \frac{G_\beta}{\sqrt{2}}\left[\bar{\psi}_p\gamma_\mu(1-g_A\gamma_5)\psi_n\right]\left[\bar{\psi}_e\gamma^\mu(1-\gamma_5)\psi_\nu\right] + h.c., \tag{2.27}$$

where the Fermi constant G_β is given by (2.19) and $g_A = -G_A/G_V \simeq 1.26$.

2.1.3 *Lepton Current Universality*

The discovery of muon neutrino as a new lepton distinct from an electron neutrino led to the concept of $\mu - e$ universality. A muon decays into an electron as $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ with its life-time $\tau_\mu = 2.2 \times 10^{-6}$sec. At first sight, μ^- could decay electromagnetically into e^- by emitting a photon as $\mu^- \to e^- + \gamma$, because this process is kinematically allowed. However, there has been no observation for this process so far.§ Existence of the particular decay mode $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ and non-existence of the decay mode $\mu^- \to e^- + \gamma$ led that in Nature there are additive conserved lepton numbers: the electron number L_e ($L_e = +1$ for e^-, ν_e and $L_e = -1$ for e^+, $\bar{\nu}_e$) and the muon number L_μ ($L_\mu = +1$ for μ^-, ν_μ and $L_\mu = -1$ for μ^+, $\bar{\nu}_\mu$). e^- interacts with ν_e alone and μ^- interacts with ν_μ alone. Later, the third lepton members, the tau τ and its neutrino ν_τ (tau neutrino) were also dicsovered with additive conserved tau number L_τ ($L_\tau = +1$ for τ^-, ν_τ and $L_\tau = -1$ for τ^+, $\bar{\nu}_\tau$). Now we have three families of leptons as

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \quad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}. \tag{2.28}$$

The universality of weak interactions is expressed by writing the total weak current of leptons as a sum of an electron, a muon and a tau current with equal weight as,

$$\begin{aligned} J_\mu^{(\ell)} &= J_\mu^{(e)} + J_\mu^{(\mu)} + J_\mu^{(\tau)} \\ &= \bar{\nu}_e \gamma_\mu (1 - \gamma_5) e + \bar{\nu}_\mu \gamma_\mu (1 - \gamma_5) \mu + \bar{\nu}_\tau \gamma_\mu (1 - \gamma_5) \tau, \end{aligned} \tag{2.29}$$

which can be rewritten as

$$J_\mu^{(\ell)} = (\bar{\nu}_e \ \bar{\nu}_\mu \ \bar{\nu}_\tau) \gamma_\mu (1 - \gamma_5) V \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}, \tag{2.30}$$

where the wave function of each particle is written by the symbol representing the particle itself and V is a 3×3 diagonal matrix given as follows;

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.31}$$

The explicit form of V shows the lepton universality.

Now let us concentrate on the muon decay $\mu^-(p_\mu) \to e^-(p_e) + \bar{\nu}_e(k_{\nu_e}) + \nu_\mu(k_{\nu_\mu})$, where momentum of each particle is given in parenthesis. The decay amplitude is given by

$$M = \frac{G_\mu}{\sqrt{2}} \left[\bar{u}(k_{\nu_\mu}) \gamma_\rho (1 - \gamma_5) u(p_\mu) \right] \left[\bar{u}(p_e) \gamma^\rho (1 - \gamma_5) v(k_{\nu_e}) \right], \tag{2.32}$$

where u and v denote the spinors for particles and anti-particles, respectively. The weak coupling constant G_μ can be determined from measured value of the life-time of muon. Since the muon decay involves no complication due to hadronic currents, we can accurately determine the value of G_μ.

§Only upper bound of the branching ratio for this decay mode, being extremely small, is known: $Br(\mu \to e + \gamma) < 1.2 \times 10^{-11}$.

The decay rate (see (B.15) is given by

$$d\Gamma = \frac{1}{2E_\mu}\overline{|M|^2}dR, \tag{2.33}$$

where dR is the Lorentz invariant phase space (see (B.14) given by

$$\begin{aligned} dR &= \frac{d^3p_e}{(2\pi)^3 2E_e}\frac{d^3k_{\nu_e}}{(2\pi)^3 2\omega_{\nu_e}}\frac{d^3k_{\nu_\mu}}{(2\pi)^3 2\omega_{\nu_\mu}}(2\pi)^4\delta^{(4)}(p_\mu - p_e - k_{\nu_\mu} - k_{\nu_e}) \\ &= \frac{1}{(2\pi)^5}\frac{d^3p_e}{2E_e}\frac{d^3k_{\nu_e}}{2\omega_{\nu_e}}\theta(E_\mu - E_e - \omega_{\nu_e})\delta((p_\mu - p_e - k_{\nu_e})^2). \end{aligned} \tag{2.34}$$

In (2.34), θ is the step function, $\theta(x) = 1$ for $x > 0$ and $= 0$ for $x < 0$. $\overline{|M|^2}$ denotes the probability of spin-averaged for initial particles and spin-summed for final particles,

$$\begin{aligned} \overline{|M|^2} &= \frac{1}{2}\sum_{\text{spin}}|M|^2 \\ &= \frac{1}{2}\frac{G_\mu^2}{2}\sum_{\text{spin}}\left[\bar{u}(k_{\nu_\mu})\gamma_\rho(1-\gamma_5)u(p_\mu)\bar{u}(p_\mu)\gamma_\sigma(1-\gamma_5)u(k_{\nu_\mu})\right] \\ &\times \sum_{\text{spin}}\left[\bar{u}(p_e)\gamma^\rho(1-\gamma_5)v(k_{\nu_e})\bar{v}(k_{\nu_e})\gamma^\sigma(1-\gamma_5)u(p_e)\right] \\ &= \frac{G_\mu^2}{4}\mathrm{Tr}\left[\not{k}_{\nu_\mu}\gamma_\rho(1-\gamma_5)(\not{p}_\mu - m_\mu)\gamma_\sigma(1-\gamma_5)\right] \\ &\times \mathrm{Tr}\left[\not{p}_e\gamma^\rho(1-\gamma_5)\not{k}_{\nu_e}\gamma^\sigma(1-\gamma_5)\right] \\ &= 64G_\mu^2(k_{\nu_e}\cdot p_\mu)(k_{\nu_\mu}\cdot p_e), \end{aligned} \tag{2.35}$$

where the electron mass was neglected because it is very small as $m_e \le m_\mu/200$. In the rest frame of muon $p = (m_\mu, 0, 0, 0)$, we have $(k_{\nu_e}\cdot p_\mu)(k_{\nu_\mu}\cdot p_e) = \frac{1}{2}\omega_{\nu_e}m_\mu(m_\mu^2 - 2m_\mu\omega_e)$ and thus we can write the decay rate as

$$\begin{aligned} d\Gamma &= \frac{G_\mu^2}{2m_\mu\pi^5}\frac{d^3p_e}{2E_e}\frac{d^3k_{\nu_e}}{2\omega_{\nu_e}}m_\mu\omega_{\nu_e}(m_\mu^2 - 2m_\mu\omega_{\nu_e}) \\ &\times \delta(m_\mu^2 - 2m_\mu E_e - 2m_\mu\omega_{\nu_e} + 2E_e\omega_{\nu_e}(1-\cos\theta)), \end{aligned} \tag{2.36}$$

where θ is an angle between produced electron and anti-neutrino. Using the relation, $d^3p_e d^3k_{\nu_e} = 4\pi E_e^2 dE_e \cdot 2\pi\omega_{\nu_e}^2 d\omega_{\nu_e} d\cos\theta$, and integrating over $\cos\theta$ using δ function, we obtain

$$d\Gamma = \frac{G_\mu^2}{2\pi^3}dE_e d\omega_{\nu_e}\omega_{\nu_e}(m_\mu^2 - 2m_\mu\omega_{\nu_e}). \tag{2.37}$$

From the condition $-1 \le \cos\theta \le 1$, the energy ω_{ν_e} and E_e are constarined as $m_\mu/2 - E_e \le \omega_{\nu_e} \le m_\mu/2$ and $0 \le E_e \le m_\mu/2$, respectively. Then, the energy spectrum of electron is obtained by integrating ω_{ν_e} over this region

$$\frac{d\Gamma}{dE_e} = \frac{G_\mu^2 m_\mu^2 E_e^2}{4\pi^3}\left(1 - \frac{4}{3}\frac{E_e}{m_\mu}\right). \tag{2.38}$$

The total muon decay rate is obtained by integrating by E_e

$$\Gamma = \frac{1}{\tau_\mu} = \frac{G_\mu^2 m_\mu^2}{4\pi^3}\int_0^{E_e^{max}} dE_e E_e^2\left(1 - \frac{4}{3}\frac{E_e}{m_\mu}\right) = \frac{G_\mu^2 m_\mu^5}{192\pi^3}, \tag{2.39}$$

where the maximum of electron energy is $E_e^{max} = m_\mu/2$. Using $\tau_\mu \cong 2.2 \times 10^{-6}$sec and $m_\mu \cong$ 105 MeV, we get

$$G_\mu = 1.166 \times 10^{-5}\mathrm{GeV}^{-2} \approx 10^{-5}/m_p^2. \tag{2.40}$$

Comparing (2.40) with (2.19), we see that G_μ for muon decays is almost equal to G_β for nuclear β-decays and thus the weak coupling constant looks universal. Therefore, the origin of the weak interaction seems to be same. In fact, by taking account of radiative corrections (Roos and Sirlin 1971; Beg and Sirlin, 1982), we see that the values of G_μ agrees with the one of G_β within only about 2%. However, it is very important to notice this small difference between couplings in lepton currents and hadron currents, which led Cabibbo to the splended idea of the so-called Cabbibo mixing currents discussed later.

Before coming to the discussion of Cabibbo currents, we just consider the decay of the third lepton τ. The decay width of τ is similarly calculated as in the muon case discussed above. This can be done by just replacing m_μ and G_μ by m_τ and G_τ, respectively. Then, using the relation

$$\tau_\tau = \tau_\mu(\frac{G_\mu}{G_\tau})^2(\frac{m_\mu}{m_\tau})^5\frac{Br(\tau^- \to e^-\bar{\nu}_e\nu_\tau)}{Br(\mu^- \to e^-\bar{\nu}_e\nu_\mu)}, \tag{2.41}$$

and the experimental data of m_τ = 1776.99 MeV, m_μ = 105.658357MeV, τ_μ = 2.19703×10^{-6}sec, $\tau_\tau = 2.906\times10^{-13}$sec, $Br(\tau^- \to e^-\bar{\nu}_e\nu_\tau) = 0.1784$ and $Br(\mu^- \to e^-\bar{\nu}_e\nu_\mu)$ = 100%, we can obtain $G_\tau/G_\mu = 1.001$. The result confirms again the universality of the weak interaction.

In summary, the near equality of coupling constant involved in the weak leptonic decay of μ, τ leptons and in the β-decay, i.e. $G_\mu \approx G_\tau \approx G_\beta$ shows that there is a universal weak interaction constant G in all weak processes. In other words, the value of weak charge is universal and therefore, it is customary to represent this universal coupling constant G by G_F called the Fermi constant.

2.1.4 *Pion decays*

Can we understand the meson decays, too, by the $V - A$ interaction? the answer is YES. We can explain them also by the $V - A$ interaction. Let us consider the pion decay as an example. A pion can decay into a muon($Br \approx 100\%$) or an electron($Br = 1.230 \times 10^{-4}$) as

$$\begin{aligned} \pi^- &\to \mu^- + \bar{\nu}_\mu, &(2.42)\\ &\to e^- + \bar{\nu}_e. &(2.43)\end{aligned}$$

For these decays, there are two remarkable experimental findings; (1)the observed μ^- in the process (2.42) was right-handed. This observation results in the conclusion that the $\bar{\nu}_\mu$ is also right-handed, since in the rest frame of π^- being spinless, μ^- and $\bar{\nu}_\mu$ run into opposite direction with equal momentum and because of angular momentum conservation, two decay particles must be right-handed. This result is cosisitent with the $(1-\gamma_5)$ factor in the lepton current. (2)The observed ratio of branching ratios of (2.43) to (2.42) is extremely small, i.e.

$$\frac{\Gamma(\pi^- \to e^- + \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- + \bar{\nu}_\mu)} \approx 1.23 \times 10^{-4}. \tag{2.44}$$

This is completely contradictory to the prediction of the phase space effect. The origin of the strong suppression must be dynamical. Actually, this is due to the $V-A$ structure of the lepton current. In fact, since $\bar{\nu}_e$ is right-handed, the emitted electron must be also right-handed because of angular momentum conservation. But this is the "wrong" helicity state for light electron, because a fast moving electron should be left-handed as described in 2.1.2. This is not the case for a muon; a muon emitted in the pion decay is rather heavy and has the right-handed component, i.e. the "right" helicity state.

Let us go to a quantitative discussion. Assuming the $V-A$ interaction, the matrix element for these decays $\pi^-(q) \to \ell^-(p) + \bar{\nu}_\ell(k)$ where $\ell = \mu$ or e and momenta of $\pi^-, \ell^-, \bar{\nu}_\ell$ are given in parentheses, is written by

$$M = \frac{G}{\sqrt{2}} < 0|J_\mu^{\pi^-}|\pi^-(q) > \bar{u}_\ell(p)\gamma^\mu(1-\gamma_5)v_{\nu_\ell}(k), \tag{2.45}$$

where $< 0|J_\mu^{\pi^-}|\pi^-(q) >$ contains all strong interaction effects, being called the pion decay constant f_π, expressing the effect of the pion decay into vacuum, and must be vector or axialvector so that this factor can contract with the lepton current to make M be Lorentz scalar. Since a π^- is spinless, available 4-vector is q alone. Then we can write

$$< 0|J_\mu^{\pi^-}|\pi^-(q) >= q_\mu f_\pi. \tag{2.46}$$

Substituting (2.46) into (2.45) and using $q = p + k$, we can write the amplitude as

$$\begin{aligned} M &= \frac{G}{\sqrt{2}}(p_\mu + k_\mu) f_\pi \left[\bar{u}_\ell(p)\gamma^\mu(1-\gamma_5)v_{\nu_\ell}(k)\right], \\ &= \frac{G}{\sqrt{2}} m_\ell f_\pi \bar{u}_\ell(p)(1-\gamma_5)v_{\nu_\ell}(k), &(2.47)\end{aligned}$$

where in going from the 1st line to the 2nd line, we used the Dirac equation for a lepton ℓ with mass m_ℓ and the one for a neutrino ν_ℓ with zero mass. Then, using the standard technique taken in the calculation of the muon decay width described in the previous section, we obtain the result

$$\Gamma(\pi^- \to \ell^- + \bar{\nu}_\ell) = \frac{G^2}{8\pi} f_\pi^2 m_\pi m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2}\right)^2 . \tag{2.48}$$

Then, we have the ratio

$$\frac{\Gamma(\pi^- \to e^- + \bar{\nu}_e)}{\Gamma(\pi^- \to \mu^- + \bar{\nu}_\mu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi - m_\mu^2}\right)^2 \approx 1.28 \times 10^{-4}, \tag{2.49}$$

which fantasitcally agrees with the experimental result of (2.44). Furthermore, from the observed life-time of π^-, $\tau = 2.6033 \times 10^{-8}$sec, we can estimate the value of the decay constant f_π as

$$f_\pi \simeq 0.92 m_\pi \tag{2.50}$$

which is an order of the pion mass.

Above observation suggests us that the $V - A$ interaction works well even for meson decays.

2.1.5 *Cabibbo Currents*

The universal Fermi interaction of the $V - A$ current–current form is quite successful in describing observed leptonic weak decays. However, the situation is expected to be less clear in the treatment of semileptonic and hadronic weak decays because of hadronic strong interaction effects. There are many such weak decay transitions of hadrons including strangeness. Experiments show that the strangeness non-conserving ($\Delta S \neq 0$) weak decays are relatively suppressed as compared to the strangeness conserving ($\Delta S = 0$) weak decays.

A modification of the weak current that explains the observed suppression of $\Delta S = 1$ transitions compared to $\Delta S = 0$ transitions, was proposed by Cabibbo in 1963 (Cabibbo, 1963). Instead of introducing new couplings to accommodate strange particle decays, he tried to keep universality by modifying the hadronic current. He assumed that the total weak current of hadrons flows into the $\Delta S = 0$ and $\Delta S = 1$ branches, keeping the total weak current of hadrons

$$J_\mu^{(h)} = a J_\mu^{(0)} + b J_\mu^{(1)}, \tag{2.51}$$

where $J_\mu^{(0)}$ and $J_\mu^{(1)}$ are currents corresponding to $\Delta S = 0$ and $\Delta S = 1$ transitions, respectively. $J_\mu^{(0)}$ and $J_\mu^{(1)}$ are normalized so that the strength of the corresponding transitions are given by the coefficients a and b. Assuming that the total weak hadronic current remains unchanged, we have

$$|a|^2 + |b|^2 = 1. \tag{2.52}$$

It is customary to write $a = \cos\theta_c$ and $b = \sin\theta_c$, where θ_c is called Cabibbo angle. The normalization condition (2.52) is then automatically satisfied and the weak current of hadrons is written by a mixing of the strangeness conserving ($\Delta S = 0$) current and the strangeness changing ($\Delta S = 1$) current

$$J_\mu^{(h)} = \cos\theta_c\ J_\mu^{(0)} + \sin\theta_c\ J_\mu^{(1)}. \tag{2.53}$$

To find the value of the Cabibbo angle θ_c, we compare the decay width of $\pi^- \to \mu^- + \nu_\mu$ and $K^- \to \mu^- + \nu_\mu$. The decay width for $\pi^- \to \mu^- + \nu_\mu$ is given by (2.48) in which G is replaced by $\cos\theta_c G$ by taking account of the Cabibbo mixing effect presented in (2.53). The decay width for $K^- \to \mu^- + \nu_\mu$ is also given by replacing m_π, f_π and G by m_K, f_K and $\sin\theta_c G$, respectively. Then, we have

$$\frac{\Gamma(K^- \to \mu^- + \nu_\mu)}{\Gamma(\pi^- \to \mu^- + \nu_\mu)} = \frac{\sin\theta_c^2}{\cos\theta_c^2}\frac{f_K^2}{f_\pi^2}\frac{m_K\left(1 - \frac{m_\mu^2}{m_K^2}\right)}{m_\pi\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)}. \tag{2.54}$$

Assuming the $SU(3)$ symmetry for meson decay constants of $f_K = f_\pi$ and using the experimental result(≈ 1.335) for the lhs of (2.54) and the values of m_μ, m_π, m_K presented in the particle data table (Particle Data Group, 2002), we obtain

$$\tan\theta_c \simeq 0.275, \tag{2.55}$$

which leads to $\sin\theta_c \approx 0.26$. More detailed analyses were carried out with the results, $\sin\theta_c = 0.220 \pm 0.002$ from $K^- \to \pi^0 + e^- + \bar{\nu}_e$ (Braun et al., 1975) and $\sin\theta_c = 0.231 \pm 0.003$ from $\Lambda \to p + e^- + \bar{\nu}_e$ (Bourquin et al., 1983). Because of the small values of θ_c, those decays whose amplitudes are proportional to $\cos\theta_c$, are known as Cabibbo-favored decays, while those with amplitude proportional to $\sin\theta_c$ are Cabibbo-suppressed decays.

Let us now understand Cabibbo's proposal at the quark level. The β decay process $n \to pe^-\nu_e$ at a quark level is $d \to ue^-\nu_e$ (Fig.2-4 (a)): one of the d quarks in the neutron (ddu) transforms into a u quark with remaining d and u quarks as spectators. In contrast, in the decay process $\Lambda^0 \to pe^-\bar{\nu}_e$ in which Λ^0 has quark content uds, the strange quark in Λ^0 transforms into a u quark (Fig.2-4 (b)). Again this involves charge changing weak currents but in this case there is also a change of strangeness at the baryon vertex and hadronic current is therefore called strangeness changing or $\Delta S = 1$ weak current. The quark current has the same $V - A$ structure

$$J_\mu^{(q\to q')} = \bar{q}'\gamma_\mu(1 - \gamma_5)q. \tag{2.56}$$

Furthermore, we retain the universality for all particles such as quarks as well as leptons and use the same coupling constant for $\Delta S = 1$ and $\Delta S = 0$ decays, except for the Cabibbo angle. In weak interactions with charged curents, leptons can only

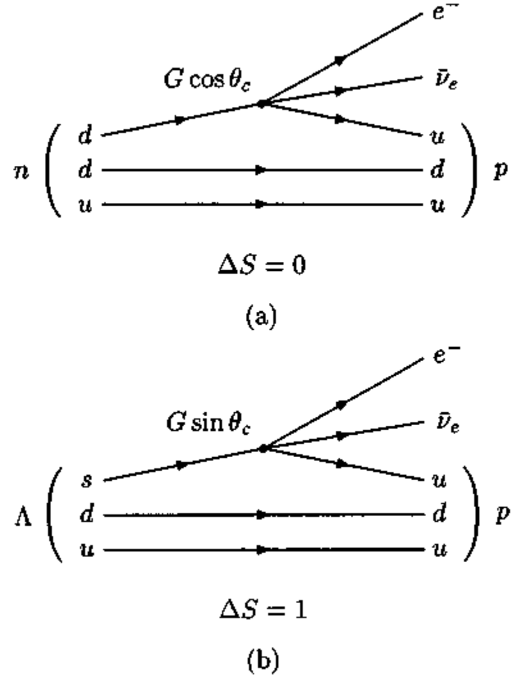


Fig. 2.4

be transformed into their partners in the same doublet; e.g. $e^- \leftrightarrow \nu_e$ for $\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$ and $\mu^- \leftrightarrow \nu_\mu$ for $\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}$. Similarly, we can group the quarks into families like $\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} c \\ s \end{pmatrix}$. But for quark transitions, they are observed not only within a family but, to a lesser degree, from one family to another via Cabibbo mixing. Therefore, for charged currents of quarks, the "partner" of the flavor eigenstate u is not just the flavor eigenstate d but a linear combination of d and s, which is called d'. Similarly, the "partner" of the c quark is another linear combination of d and s which is orthogonal to d' and is called s'. The coefficients of these linear combinations can be written by using a single mixing parameter θ_c as

$$\begin{aligned} d' &= \cos\theta_c\, d + \sin\theta_c\, s, \\ s' &= -\sin\theta_c\, d + \cos\theta_c\, s. \end{aligned} \tag{2.57}$$

(2.57) can be written in the matrix form as

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = U \begin{pmatrix} d \\ s \end{pmatrix}, \quad \text{with} \quad U = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}. \tag{2.58}$$

U is unitary and performs the rotation of quark states, $\begin{pmatrix} u \\ d \end{pmatrix}$ and $\begin{pmatrix} c \\ s \end{pmatrix}$, into "rotated" states, $\begin{pmatrix} u \\ d' \end{pmatrix}$ and $\begin{pmatrix} c \\ s' \end{pmatrix}$. The transitions $s \leftrightarrow u$ as compared to $d \leftrightarrow u$ are therefore suppressed by a factor of $\sin^2\theta_c : \cos^2\theta_c \approx 1 : 20$, in accordance with the data.

If we add the third generation of quarks $\begin{pmatrix} t \\ b \end{pmatrix}$, the 2×2 matrix of (2.58) is replaced by a 3×3 matrix which was originally introduced by Kobayashi and Maskawa in 1973 (Kobayashi and Maskawa, 1973) and is called the Cabibbo–Kobayashi–Maskawa matrix (CKM matrix),

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} U_{ud} & U_{us} & U_{ub} \\ U_{cd} & U_{cs} & U_{cb} \\ U_{td} & U_{ts} & U_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{2.59}$$

The probability for a transitions from a quark q to another quark q' is proportional to $|U_{qq'}|^2$, the square of the matrix element. The diagonal elements of this matrix describe transitions within a family; they deviate from unity by only a few percent. It was found from the analysis of various meson decay data that the values of matrix elements U_{us} and U_{cd} were nearly one order of magnitude smaller than those of U_{ud} and U_{cs}. Furthermore, transitions from the third to the second generation ($t \to s$, $b \to c$) are further suppressed compared to transitions from the second to the first generation. The average values of $|U_{ud}|$, $|U_{us}|$, $|U_{cd}|$ and $|U_{cs}|$ obtained from the analysis of experimental data (Particle Data Group, 2002) are

$$\begin{aligned} |U_{ud}| &= 0.9735 \pm 0.00108, \quad |U_{us}| = 0.2196 \pm 0.0023, \\ |U_{cd}| &= 0.224 \pm 0.016, \quad |U_{cs}| = 1.04 \pm 0.16. \end{aligned} \tag{2.60}$$

The observation that strangeness non-conserving weak interactions are relatively suppressed as compared to the strangeness conserving weak interactions led to further extension of the concept of universality to involve weak hadronic currents. After all, the Cabibbo theory established the quark–letpon universality and sloved the puzzle on the slight discrepancy between the value of G_β (in (2.19)) determined from the nuclear β-decay and the one of G_μ (in (2.40)) from the μ-decay; in fact, $G_\beta = G_\mu \cos\theta_c$ with $\cos\theta_c \approx 0.975$.

2.1.6 *Difficulties in the Fermi Theory*

As we have seen in the previous sections, the Fermi theory of weak interactions, whose definitive form was established by Feynman and Gell-Mann as the current–current form with $V - A$ currents and which was extended by Cabibbo to processes including hadrons such as $\Delta S = 0$ and $\Delta S = 1$ decays, works very well in describing

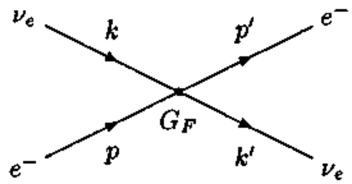


Fig. 2.5

phenomena of the observed charged current weak interactions. The Fermi theory of its simple and elegant form could not only explain many experimental data of decays but also matches well with the physical principle such as universality, algebraic properties embodied in Cabibbo currents, and so on.

However, it is now also well known that the Fermi theory cannot be a complete theory, even if it works well as phenomenology. When we apply it to scattering processes, the lowest order approximation violates the unitarity bound. Furthermore, unlike the quantum electrodynamics for electromagnetic interactions, the Fermi interaction is not a renormalizable theory and thus we cannot manage those higher-order contributions to yield a finite outcome.

2.1.7 *Unitarity violation*

To see how the Fermi interaction violates the unitarity bound, let us consider the neutrino–electron scattering (see Fig. 2.5)

$$\nu_e(k) + e^-(p) \to e^-(p') + \nu_e(k'), \tag{2.61}$$

whose amplitude is given by

$$M = \frac{G_F}{\sqrt{2}} \left[\bar{u}_{\nu_e}(k')\gamma_\mu(1-\gamma_5)u_e(p)\right]\left[\bar{u}_e(p')\gamma^\mu(1-\gamma_5)u_{\nu_e}(k)\right], \tag{2.62}$$

from which, by neglecting the electron mass m_e and following the standard technique for calculating cross sections, we can obtain the differential cross section as

$$\frac{d\sigma}{d\Omega} = \frac{\overline{|M|^2}}{64\pi^2 s} = \frac{G_F^2}{4\pi^2}s. \tag{2.63}$$

where $s = (k+p)^2 = 4E^2 = 4k^2$ ($E(k)$ is the CMS energy(momentum) of the $\nu_e + e^-$ system). Integrating over the solid angle Ω, we obtain the total cross section for this process

$$\sigma = \frac{G_F^2}{\pi}s. \tag{2.64}$$

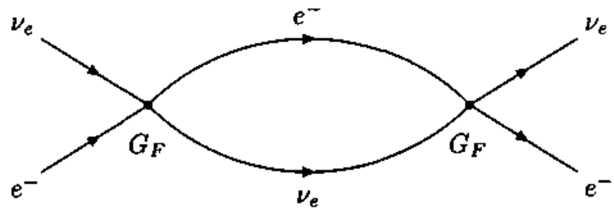


Fig. 2.6

On the other hand, as is well known, every scattering cross section can be decomposed into partial waves. For spinless particles, we have in general

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left| \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell + 1) f_\ell P_\ell(\cos\theta) \right|^2 , \tag{2.65}$$

where f_ℓ is the partial wave amplitude for an orbital angular momentum ℓ. For the pointlike Fermi interaction, only f_0 contributes to $\nu_e - e^-$ scattering (2.61) and there is no angular dependence (with $P_0(\cos\theta) = 1$). Then, we expect to have

$$\frac{d\sigma}{d\Omega} = \frac{1}{4E^2} |f_0|^2. \tag{2.66}$$

However, since unitarity requires $|f_\ell| \leq 1$ for every partial wave, we obtain the upper bound for the diffrential cros section as

$$\frac{d\sigma}{d\Omega} \leq \frac{1}{4E^2}. \tag{2.67}$$

Then, the total cross section is also bounded similarly as

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = 4\pi \frac{d\sigma}{d\Omega} \leq \frac{\pi}{E^2}. \tag{2.68}$$

Therefore, the prediction of (2.64), being in the lowest order of the Fermi interaction, violates unitarity bound at some high energy

$$E \geq \frac{1}{\sqrt{2}} \sqrt{\frac{\pi}{G_F}} \approx 370\text{GeV}. \tag{2.69}$$

One might suppose that this is due to neglection of higher-order corrections of Fermi interactions. But this is not the case. For example, if we take into account the 2nd order diagram shown in Fig. 2.6, we will find that the calculation leads to a divergent result. We cannot cure this disease from those higher order corrections. Rather, we can see that the root lies in the Fermi interaction itself.

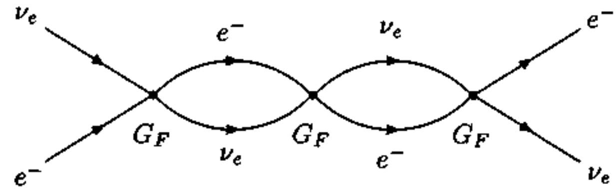


Fig. 2.7

2.1.8 *Non-renormalizability*

Another serious difficulty in the Fermi theory is that it is not a renormalizable theory, which is very different from the case of quantum electrodynamics(QED), being renormalizable. As is well known, we have divergent integrals originating from higher-order diagrams even in QED as well. However, in QED we can remove those divergences by renormalizing the mass and charge into the observed physical values. On the other hand, in the case of the Fermi interaction, if we try to follow the renormalization procedure analogously as in QED and go to the next higher order diagram, we will encounter a more serious divergence, which needs further renormalization constants. Then, when we consider all higher order diagrams, we need an infinite set of renormalization constants. That is, the theory cannot be renormalized. (see, for example, Aitchison and Hey, 1989)

By taking the reaction $\nu_e + e^- \to \nu_e + e^-$, let us see what is going on. The lowest order diagram for this reaction is given in Fig. 2.5, which contributes to the amplitude with the order of G_F. The 2nd order correction comes from Fig. 2.6. There we have two fermion propagators, each one behaves as $1/k$ for large internal fermion momentum k, and thus due to these internal fermion propagators we have a loop integral contribution to the amplitude with $G_F^2 \int \frac{d^4k}{k^2}$, which is divergent. Similarly, the 3rd order diagram (Fig. 2.7) contributes to the amplitude with $G_F^3 (\int \frac{d^4k}{k^2})^2$, which is more seriously divergent. After all, we have a series of expansion with $G_F \int \frac{d^4k}{k^2}$, where the higher order terms yield more and more severe divergence. This is because G_F has a dimension of $[\text{mass}]^{-2}$; each term has a new G_F in the expansion series, which must be compensated by a divergent factor $\int \frac{d^4k}{k^2}$ with $[\text{mass}]^2$ dimension in each term to keep the dimension of the amplidude.

2.2 Intermediate Weak Boson Model

One of the interesting ideas to attempt to take away those difficulties is the intermediate weak boson exchange model. The weak coupling constant $G(= G_F)$ appearing, for instance, in (2.27), (2.32) and (2.62) has a dimension of $[\text{mass}]^{-2}$ (see (2.40)). This fact suggest us that we can develop further an analogy between the weak interaction and the electromagnetic interaction by postulating that the

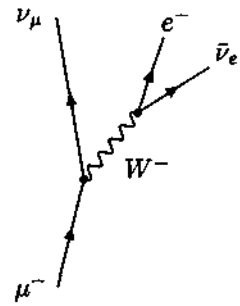


Fig. 2.8

weak interaction is mediated by a massive weak boson, just as the electromagnetic interaction is mediated by a massless photon. For example, assuming the W boson exchange between two charged lepton currents (Fig. 2.8), let us write the decay amplitude for $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ as

$$M = -\left[\frac{g}{\sqrt{2}}\bar{u}_{\nu_\mu}\gamma_\rho\frac{1-\gamma_5}{2}u_\mu\right]\frac{-g^{\rho\sigma}+\frac{q^\rho q^\sigma}{M_W^2}}{q^2-M_W^2}\left[\frac{g}{\sqrt{2}}\bar{u}_e\gamma_\sigma\frac{1-\gamma_5}{2}u_{\nu_e}\right], \qquad (2.70)$$

instead of (2.32), where g is the dimensionless coupling constant and q represents the momentum of an intermediate W boson with mass M_W. This form is quite analogous to the electron–proton scattering amplitude mediated by a massless photon

$$M = -\left[e\bar{u}_p\gamma_\mu u_p\right]\frac{-1}{q^2}\left[-e\bar{u}_e\gamma^\mu u_e\right]. \qquad (2.71)$$

We see that a photon propagator between two electromagnetic currents with charge e is replaced by a W-boson propagator between two charged currents with coupling constant $\frac{g}{\sqrt{2}}$. Since the Fermi interaction works as a contact interaction at zero range, the W boson must be very heavy, unlike a massless photon. Furthermore, at low energy where q^2 is much smaller than M_W^2, the Fermi interaction works very well and thus (2.70) should reduce to (2.32) with

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}, \qquad (2.72)$$

From (2.72), the fact that the weak interaction is weak with small G_F is considered to be not because the coupling g is small, but because M_W is large with $g \approx e$. This suggests that the electromagnetic and weak interaction might be formulated into a unified theory. In fact, later this idea was beautifully realized in the non-Abelian Gauge theory called Glashow–Weinberg–Salam(GWS) theory, which is the one of main subjects of this textbook and will be discussed in detail in the next Chapters.

But before coming into the GWS theory right now, it is instructive to learn a few things about the intermediate weak boson model. One is related to the problem

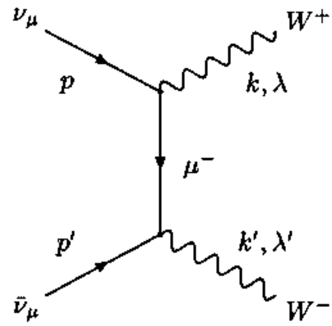


Fig. 2.9

of unitarity bound. As discussed above, the Fermi interaction, i.e. the contact interaction leads to violation of unitarity bound. Then, can the intermediate weak boson model cure this disease? The answer is NO. For example, let us consider the process, in the lowest order, (Fig. 2.9)

$$\nu_\mu(p) + \bar{\nu}_\mu(p') \to W^+(k, \lambda) + W^-(k', \lambda'), \tag{2.73}$$

which is predicted to exist in the intermediate weak boson model. In (2.73), the momentum of each particle is given in parentheses, and λ and λ' denote the polarization of W^+ and W^-, respectively. The invariant amplitude for this process is given by

$$M^{(\lambda,\lambda')} = (\frac{g}{\sqrt{2}})^2 \varepsilon_\mu^{*(\lambda')}(k')\varepsilon_\nu^{*(\lambda)}(k)\bar{v}(p')\gamma^\mu \frac{1-\gamma_5}{2} \frac{(\not{p} - \not{k} + m_\mu)}{(p-k)^2 - m_\mu^2}\gamma^\nu \frac{1-\gamma_5}{2} u(p), \tag{2.74}$$

where $\varepsilon^{(\lambda)}(k)$ and $\varepsilon^{(\lambda')}(k')$ are the polarization vector of W^+ and W^-, respectively. From this amplitude, one can calculate the differential cross section at high energy as

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2}{8\pi^2} \sin^2\theta E^2, \tag{2.75}$$

where E and θ are the energy and the scattering angle of the incoming ν_μ in the CMS, respectively. Integration over the solid angle results in the total cross section

$$\sigma = \frac{G_F^2}{12\pi} s, \tag{2.76}$$

where $s = 4E^2$. The cross section increases with s and hence violates the unitarity bound at some high energy. Therefore, the disease of unitarity violation cannot be cured even in the intermediate weak boson model.

Another difficulty, i.e. the non-renormalizability of the Fermi interaction also cannot be solved by the intermediate weak boson model. To see this apparently, let us consider an example of scattering presented in Fig. 2.10. In this figure, (a) is the

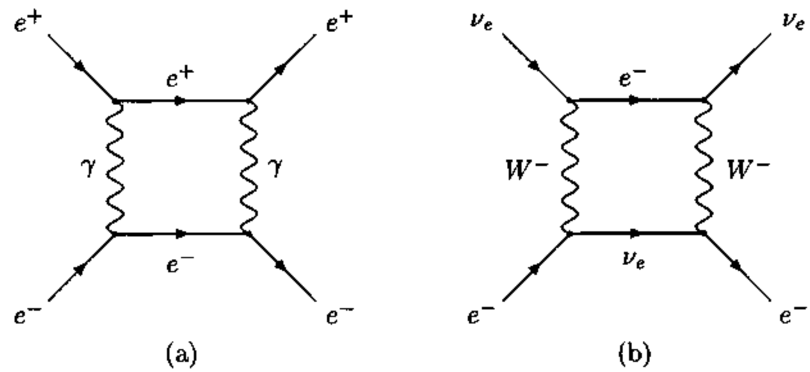


Fig. 2.10

two photon exchange correction to the QED process, $e^+ + e^- \to e^+ + e^-$, while (b) is the two W boson exchange correction to the weak process, $\nu_e + e^- \to \nu_e + e^-$. For the photon exchange diagram (a), since the photon propagator behaves as $1/k^2$ for large internal momentum k, we have a loop integral $\int \frac{d^4k}{k^6}$ for this amplitude, which is finite. On the other hand, for the W boson exchange diagram (b), since the W boson is massive and its propagator has a longitudinal component

$$\frac{-1}{k^2 - M_W^2}(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_W^2}), \tag{2.77}$$

in which k dependence vanishes at large k, the corresponding loop integral becomes $\int \frac{d^4k}{k^2}$, which is quadratically divergent and the situation does not change from the case of the Fermi interaction, i.e. the contact interaction. After all, the intermediate weak boson model cannot be renormalized to get a finite result.

After a long struggle, an extremely elegant and consistent theory without theoretical dificulties of unitarity violation and non-renormalizability was discovered for the lepton family by Weinberg and Salam in 1967 (Weinberg, 1967; Salam, 1968). The theory was extented to quark families successfully and, combined with quantum chromodynamics(QCD) which is the fundamental field theory of strong interactions, was developed as the standard model of the particle physics. But before going into the Glashow–Weiberg–Salam(GWS) theory, we need some preparations, which will be discussed in the next Chapter.

Problems

2.1 Show that the coupling constant G_i in (2.5) must be real if the interaction Hamiltoniam (2.5) for the β-decay is time–reflection invariant.

2.2 Using (2.13), prove (2.14).

2.3 In the nonrelativistic approximation, one can see that the Fermi transition for β-decay processes occurs for $O_i = V$ and A, and the Gamow–Teller transition does for $O_i = A$ and T, while $O_i = P$ can contribute to the Fermi transition only in the order of $\frac{v}{c^2}$ and thus can be neglected. Show these results using the nonrelativistic expression for the Dirac spinor of nucleons.

2.4 Starting with (2.45), derive (2.48).

2.5 Stating with (2.62), drive (2.63).

2.6 Starting with (2.74), drive (2.75).

Chapter 3

SYMMETRIES AND THE GAUGE THEORIES

One of the most fundamental principles in particle physics is that interactions among fundamental particles are described by symmetry principles. The invariance of the Lagrangian under certain symmetry transformations leads to a set of conservation laws. In addition to space-time symmetries such as Lorentz invariance, parity invariance, time reversal invariance etc., the internal symmetries such as isospin, flavor, color etc. have been also known. The development of our understanding about the fundamental interactions is based, to a large extent, on our understanding of such underlying symmetries of Nature. It is remarkable to know that there is a connection between exact symmetries and conservation laws and to see that the requirement of local gauge invariance can serve as a dynamical principle to guide the construction of interacting field theories.

In this Chapter, we first discuss the relation of the global symmetry to the Noether's theorem, which connects the invariance of the Lagrangian under a continuous symmetry transformation to a conservation law and the conserved quantum numbers. Then the local gauge theory will be developed with an example of quantum electrodymanics(QED), being an Abelian gauge theory with $U(1)$ symmetry. A non-Abelian Yang–Mills gauge theory is also discussed with an example of $SU(2)$ symmetry. Furthermore, an idea and an important result of the spontaneous symmetry breaking(SSB) and the Goldstone theorem will be discussed. Finally, we will discuss the Higgs mechanism, which is realized in the mechanism of SSB with local gauge symmetries.

3.1 Global Symmetries and Noether's Theorem

In the field theory, fields are used to describe the fundamental particles. Dynamics among those particles are given by the Lagrangian density* $\mathcal{L}(\phi(x), \partial_\mu \phi(x))$ which depends on the field $\phi(x)$ and its derivative $\partial_\mu \phi(x)$. Fields are arranged so as to realize various symmetries mentioned above. The Lagrangian formalism provides

*The Lagrangian density is often called, simply, Lagrangian. We follow the same usage.

a systematic way of identifying these symmetries and extracting the constants of motion associated with these symmetries.

Let us consider a system composed of a set of n independent fields[†] $\phi_a(x)$ ($a = 1, 2, \cdots, n$) and let the Lagrangian of this system be invariant under the transformation of a certain group G. The equation of motion for $\phi_a(x)$, called the Euler–Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} = 0, \quad (a = 1, 2, \cdots, n), \tag{3.1}$$

is generated from the requirement of least action,

$$\delta S = \delta \int d^4x \mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x)) = 0. \tag{3.2}$$

We define the "momentum" fields which are canonically conjugate to field variables $\phi_a(x)$ as

$$\pi_a(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)}, \tag{3.3}$$

and assume finally the canonical commutation relations to quantize the system

$$[\phi_a(\vec{x}, t), \phi_b(\vec{y}, t)] = [\pi_a(\vec{x}, t), \pi_b(\vec{y}, t)] = 0, \tag{3.4}$$

$$[\phi_a(\vec{x}, t), \pi_b(\vec{y}, t)] = i\delta_{ab}\delta^3(\vec{x} - \vec{y}). \tag{3.5}$$

Now, we are interested in the variation of $\mathcal{L}$ under the transformation $\phi_a \to \phi_a + \delta\phi_a$. The result turns out

$$\begin{aligned} \delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta(\partial_\mu \phi_a) \\ &= \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right)\delta\phi_a + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right) \\ &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a\right), \end{aligned} \tag{3.6}$$

by using the Euler–Lagrange equation (3.1). When the transformation parameters are independent of the space-time point x, it is called the "global" transformation. As an example, we introduce a transformation under a special unitary group $G = SU(n)$ as

$$\delta\phi_a = -i\theta^i T^i_{ab}\phi_b(x) \qquad (\phi_a \to \phi'_a = e^{-i\theta^i T^i_{ab}}\phi_b), \tag{3.7}$$

using x-independent infinitesimal parameters θ^i and an $n \times n$ matrices T^i, where $a, b = 1, 2, \cdots, n$ and $i = 1, 2, \cdots, n^2 - 1$. Here the summation over over i and also

[†]For simplicity, we consider the case of boson fields for the moment. For the case of fermion fields, we need no change in the following discussion if commutation relations (3.4) and (3.5) below are replaced by anti-commutation relations.

over b is implied. T^i is called generators of the $SU(n)$ group. Then, when we define the current

$$j^{i\mu}(x) \equiv -i\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} T^i_{ab}\phi_b, \tag{3.8}$$

(3.6) can be written as

$$\delta\mathcal{L} = \theta^i \partial_\mu j^{i\mu}(x). \tag{3.9}$$

Thus, if the transformation $\phi_a \to \phi_a + \delta\phi_a$ leaves $\mathcal{L}$ invariant, i.e. $\delta\mathcal{L} = 0$, we have the current conservation law,

$$\partial_\mu j^{i\mu}(x) = 0. \tag{3.10}$$

By integrating this equation over all space,

$$\int d^3x \left(\frac{\partial}{\partial t} j^{i0} + \vec{\nabla}\cdot\vec{j}^i\right) = 0, \tag{3.11}$$

and by assuming for $\vec{j}^i(x)$ to vanish at spatial infinity, we are led to, by using the Gauss's theorem, the charge conservation law,

$$\frac{dQ^i}{dt} = 0, \tag{3.12}$$

where the "charge" is defined by

$$Q^i(t) \equiv \int d^3x j^{i0}(\vec{x}, t). \tag{3.13}$$

One can see that a symmetry of the Lagrangian under a field transformation implies conservation laws; this is called the Noether's theorem (Noether, 1918) and the current $j^{i\mu}(x)$ is called Noether currents.

In quantum theory, we have the Heisenberg equation

$$\partial_\mu j^{i\mu}(x) = i[P_\mu, j^{i\mu}(x)], \tag{3.14}$$

where P_μ is the four-momentum operator. If the current is conserved, one can obtain, by integrating over all space,

$$\frac{dQ^i}{dt} = i[H, Q^i] = 0, \tag{3.15}$$

with $H \equiv P_0$, which is the Hamiltonian. Therefore, it turns out that Q^i commute with Hamiltonian H and the states with Q^i ($i = 1, 2, \cdots, n^2-1$) degenerate with the same energy so as to make a multiplet. Using the canonical commutation relations, one can easily show that the charges Q^i satisfy the following relation, which is the Lie algebra,

$$[Q^i, Q^j] = if_{ijk}Q^k, \tag{3.16}$$

since the generators T^i of $SU(n)$ always satisfy the same Lie algebra,

$$[T^i, T^j] = i f_{ijk} T^k, \tag{3.17}$$

where f_{ijk} is the structure constant of the Lie group and totally antisymmetric with respect to i, j and k. That is, the charges Q^i satisfy the same Lie algebra as the generators T^i of the group $SU(n)$ do. Furthermore, for an infinitesimal $\theta^i(\ll 1)$, one can derive the following relation, using canonical commutation relations of (3.4) and (3.5),

$$[Q^i, \phi] = -T^i\phi. \tag{3.18}$$

This relation shows that Q^i is the generator of an infinitesimal transformation

$$\phi \to \phi' = e^{i\theta^i Q^i}\phi e^{-i\theta^i Q^i} = e^{-i\theta^i T^i}\phi, \tag{3.19}$$

as can be seen by comparing the following relation

$$e^{i\theta^i Q^i}\phi e^{-i\theta^i Q^i} = \phi + i\theta^i[Q^i, \phi] + \mathcal{O}(\theta^2), \tag{3.20}$$

with

$$e^{-i\theta^i T^i}\phi = \phi - i\theta^i T^i\phi + \mathcal{O}(\theta^2). \tag{3.21}$$

In summary, the Lagrangian is a fundamental object describing the dynamics among elementary particles. It has various symmetries which are represented by transformation groups. The Lagrangian should be constructed to be invariant under those transformations. The invariance of the Lagrangian under a global transformation specified by a unitary group results in a set of conserved charges Q^i(Noether's theorem) and these charges become a generator of the Lie algebra of the group.

3.2 Local Gauge Symmetries and Gauge Fields

The properties of strong, weak and electromagnetic interactions appear to be unrelated at low energies. For example, they have quite different coupling constants. However, there is a possibility that at some extremely high energy the coupling constant may converge to a single value and that interactions among elementary particles could be explained in terms of single unified field. Glashow, Weinberg and Salam made a major breakthrough along the path to unification by unifying the weak and electromagnetic interactions. The most significant step in this direction is the realization that all fundamental interactions are invariant under local gauge transformations and the hope is that gauge theories will provide a basis for a comprehensive unification of fundamental interactions.

The standard model of particle physics refers to three quantum gauge theories which describe electromagnetic, weak and strong interaction of elementary particles. All these theories are renormalizable and are based on certain symmetries, which are classified into two ways, (1) manifest or unbroken and (2) hidden or spontaneously broken. The electromagnetic interaction is unbroken local $U(1)$ symmetric, the weak interaction is spontaneously broken local $SU(2) \times U(1)$ symmetric and the strong interaction is unbroken local $SU(3)$ symmetric. The non-Abelian gauge theories of weak and strong interaction is the generalization of quantum electrodynamics(QED), which is the Abelian local $U(1)$ gauge theory.

3.2.1 *Quantum Electrodynamics–$U(1)$ model–*

Let us begin with the Lagrangian $\mathcal{L}$ describing the field of a single free fermion with mass m. In the natural unit($\hbar = c = 1$), it is expressed as

$$\mathcal{L}(x) = \bar{\psi}(x)\,(i\gamma^\mu \partial_\mu - m)\,\psi(x). \tag{3.22}$$

from which the Dirac equation is obtained by taking variation for $\bar{\psi}(x)$;

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0, \tag{3.23}$$

where $\psi(x)$ is a 4–component spinor field of the fermion at point x in four dimensional space-time and $\partial_\mu = \partial/\partial x^\mu$ $(\mu = 0,1,2,3)$. γ^μ are the 4×4 Dirac matrices and the summation over μ is implied. It is understood that m in the mass term is multiplied by a 4×4 unit matrix I.

First consider a global transformation for this system. As discussed in the previous section, it is described by a phase change of the fermion field

$$\psi'(x) = e^{-i\theta}\psi(x) = U_\theta \psi(x). \tag{3.24}$$

The Lagrangian (3.22) is invariant under this phase transformation where θ is a constant everywhere in space-time. That the Lagrangian is invariant under the global transformation means that the phase θ of the fermion field is not observable. Clearly, transformation (3.24) is unitary, i.e. $U_\theta^+ U_\theta = 1$ and it is Abelian, i.e. $U_{\theta_1} U_{\theta_2} = U_{\theta_2} U_{\theta_1}$. That is to say, transformation (3.24) is global, unitary and Abelian. This transformation is denoted by $U(1)$.

Now, let us study what happens when this global symmetry is made local. It is well known that the electromagnetic dynamics has a local gauge symmetry larger than a global symmetry. A local gauge transformation is defined as

$$\psi'(x) = e^{-i\theta(x)}\psi(x), \qquad \bar{\psi}'(x) = e^{i\theta(x)}\bar{\psi}(x), \tag{3.25}$$

by introducing a x-dependent phase parameter $\theta(x)$ varying locally from point to point in space–time. Then, the kinetic energy term is no longer invariant under this transformation because an additional term being proportional to $\partial_\mu \theta(x)$ appears from the derivative of the transformed fermion field. The new Lagrangian under this transformation becomes

$$\mathcal{L} \to \mathcal{L}' \;=\; \bar{\psi}'(x)(i\gamma^\mu \partial_\mu - m)\psi'(x)$$

$$\begin{aligned} &= \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) + \bar{\psi}(x)\gamma^\mu\psi(x)\partial_\mu\theta(x) \\ &= \mathcal{L} + j^\mu(x)\partial_\mu\theta(x), \end{aligned} \tag{3.26}$$

where $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$ is the vector current carried by the fermion. Hence, the Lagrangian is not invariant under the local transformation. Only if $\partial_\mu\theta(x) = 0$, i.e. $\theta(x)$ is independent of x, $\mathcal{L}$ is invariant. Then, if we make the following replacement in (3.22)

$$\partial_\mu \to D_\mu \equiv \partial_\mu - ieA_\mu(x), \tag{3.27}$$

by introducing some vector field A_μ, we get instead

$$\mathcal{L} \to \mathcal{L} + ej^\mu A_\mu, \tag{3.28}$$

where e is the electromagnetic charge of the fermion which is negative for an electron.

Thus, if we define a new Lagrangian by replacing ∂_μ in (3.22) by D_μ of (3.27)

$$\begin{aligned} \mathcal{L}(\psi, \bar{\psi}, A_\mu) &= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \\ &= \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + e\bar{\psi}\gamma_\mu\psi A_\mu, \end{aligned} \tag{3.29}$$

then this $\mathcal{L}$ becomes invariant under (3.25) if at the same time we make the replacement

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\theta(x) \quad \text{or} \quad \delta A_\mu = -\frac{1}{e}\partial_\mu\theta(x), \tag{3.30}$$

which precisely cancels the additional unwanted term in (3.26). D_μ defined above is called a "covariant derivative", a terminology borrowed from the general relativity.

From the above observation, one can prescribe the procedure for getting the gauge invariant Lagrangian in a little different way; starting from an original Lagrangian (3.22) which possesses a global symmetry and making the replacement (3.27), we can get a new Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi, \tag{3.31}$$

which is invariant under the local gauge transformation by requiring

$$(D_\mu\psi(x))' = e^{-i\theta(x)}D_\mu\psi(x) \quad \text{or} \quad \delta(D_\mu\psi) = -i\theta(x)D_\mu\psi(x). \tag{3.32}$$

The requirement (3.32) leads to (3.30), where $\theta(x)$ is the x-dependent infinitesimal phase parameter. Then, electromagnetic dynamics is made invariant by introducing a spin 1 vector (gauge) boson field A_μ called the photon through the covariant derivative, which is called the "minimal coupling". It is very important to know that in the gauge invariant theories the interaction between gauge bosons and particles (fermions and/or bosons) is uniquely determined only through the minimal coupling.

The Lagrangian which is invariant under local $U(1)$ gauge transformation is, therefore, given by (3.31). However, this is not the complete Lagrangian for describing the whole system. We have to add the kinetic energy term and mass term of the electromagnetic field $A_\mu(x)$, which should be also gauge invariant. As for the mass term, the local gauge invariance necessarily leads to massless photon because the mass term $\frac{m^2}{2}A_\mu A^\mu$ violates the local gauge invariance, unless $m = 0$. It is well known that the kinetic energy term of the electromagnetic field is given by $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where the electromagnetic field–strength tensor $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.33}$$

The coefficient $-\frac{1}{4}$ is necessary for the requirement that the Euler–Lagrange equation is just the Maxwell's equation. The kinetic energy term is obviously gauge invariant because $\delta F_{\mu\nu} = 0$ under the transformation (3.30) as easily checked by directly introducing (3.30) into (3.33). It is also seen in a little different way; the covariant derivative satisfies

$$[D_\mu, D_\nu]\,\psi = -ieF_{\mu\nu}\psi. \tag{3.34}$$

From (3.32), it follows that $([D_\mu, D_\nu]\,\psi)' = e^{-i\theta(x)}([D_\mu, D_\nu]\,\psi)$ and thus $F'_{\mu\nu}\psi' = e^{-i\theta(x)}F_{\mu\nu}\psi = F_{\mu\nu}\psi'$, that is, $F'_{\mu\nu} = F_{\mu\nu}$.

Therefore, the complete gauge invariant Lagrangian for the system of an electron and a photon takes the following form,

$$\mathcal{L}_{QED} = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3.35}$$

The field theory with $\mathcal{L}_{QED}$ is called quantum electrodynamics(QED). The generalization of classical electrodynamics to describe quantum effects had culminated in the development of QED as the theory of interaction of photons and electrons.

In QED, the electromagnetic interactions are mediated by massless photons. Photons are quanta of the electromagnetic fields, being massless with no charge and with spin 1, and they do not interact amongst themselves. Theory of gauge fields represents a class of theories which share and generalize characteristic properties of the Maxwell's theory of electromagnetic fields. There are two characteristics of Maxwell's theory: (1) Forces created by gauge fields, i.e. photons, are long-ranged and obey inverse square of distance as realized as Coulomb force. (2) Force is proportional to the quantum number of its source and that quantum number, i.e. the charge, is conserved.

Typical physical processes due to the electromagnetic interactions are shown in Fig. 3.1: (a) a fermion emitting or absorbing a photon, (b) annihilation of a fermion–antifermion pair to produce a photon, (c) a fermion scattered by another fermion in a two step process in which a photon emitted by the first fermion is absorbed by the second.

The principle of gauge invariance has come to be recognized as most powerful guiding light for our understanding of not only QED but also probably all interactions.

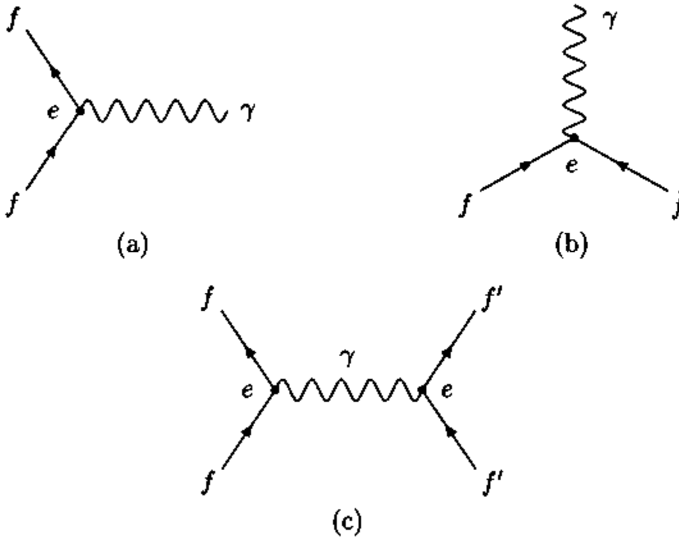


Fig. 3.1

3.2.2 *Yang–Mills Gauge Theory–SU(2) model–*

In 1954, Yang and Mills (Yang and Mills, 1954) proposed that the $U(1)$ gauge theory of electromagnetic interactions could be generalized to the non-Abelian gauge theories which are invariant under any non-commutative continuous symmetry group. They chose the isospin of $SU(2)$, which was familiar at that time. Therefore, the original Yang–Mills non-Abelian gauge theory is an extension of the $U(1)$ Abelian gauge theory of QED to the internal isospin $SU(2)$ symmetric theory. The formalism can be easily generalized to more general cases of $SU(n)$ with $n \geq 3$.

We have seen that the electromagnetic fields are described by an antisymmetric tensor $F_{\mu\nu}$, which is made of a vector potential A_μ. Now we add an isospin index to A_μ and $F_{\mu\nu}$ and ask the following question; *Under what condition is a theory invariant under a space–time coordinate and isospin dependent phase transformation?*

Likewise in (3.25), one can write an x-dependent $SU(2)$ transformation (phase rotation) as

$$\psi(x) \to \psi'(x) = U\psi(x), \tag{3.36}$$

$$U = e^{-ig\frac{\tau^i}{2}\theta^i(x)}, \tag{3.37}$$

where the summation over i is implied. g is the coupling constant associated with the group $SU(2)$. τ^i $(i = 1,2,3)$ are three 2×2 Pauli matrices and θ^i $(i = 1,2,3)$ are three real parameters corresponding to three generators $\tau^i/2$, $(i = 1,2,3)$ of the group $SU(2)$.

$$\left[\frac{\tau^i}{2}, \frac{\tau^j}{2}\right] = i\varepsilon_{ijk}\frac{\tau^k}{2} \tag{3.38}$$

where the summation over k is implied. ε_{ijk}, being the totally antisymmetric Levi–Cività symbol with $\varepsilon_{123} = 1$, is the structure constant of the group $SU(2)$. Obviously these three generators do not commute, i.e. $SU(2)$ is non-Abelian, hence the gauge theory in question is called non-Abelian. ψ is both a spinor and an isospinor. Thus, instead of a single fermion field, we start with a doublet Dirac field

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \bar{\psi} = (\bar{\psi}_1 \;\; \bar{\psi}_2),$$
$$\bar{\psi}_a = \psi_a^\dagger \gamma^0. \qquad (a = 1, 2)$$

Originally, Yang and Mills chose $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$ doublet where p and n denote the proton and neutron, respectively. However, one can also choose $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$ doublet as well where u and d denote the up-quark and down-quark, respectively. Here, the coupling strength g is a real constant to be determined from experiment (analogous to e in QED). We insist that the Lagrangian is invariant under transformation (3.36) and accordingly we encounter a number of additional complications, due to the fact that there are now three orthogonal symmetry motions which do not commute with one another. To construct a Lagrangian that is invariant under (3.36), we must again define a covariant derivative that transforms in a simple way. In analogy of QED, it is natural to introduce three independent gauge field potentials acting in different directions, namely, $A_\mu^i(x)$ for $i = 1, 2, 3$ and require that operation on ψ is only through the covariant derivative

$$D_\mu = \partial_\mu - ig\vec{\mathbf{A}}_\mu, \tag{3.39}$$

where $\vec{\mathbf{A}}_\mu = \sum_{i=1}^{3} \frac{\tau^i}{2} A_\mu^i = \frac{\vec{\tau}}{2} \cdot \vec{A}_\mu$ is a matrix-valued gauge field. Likewise in QED discussed above, the covariant derivative should satisfy

$$(D_\mu\psi)' = D'_\mu\psi' = U(D_\mu\psi), \tag{3.40}$$

which leads to

$$D'_\mu = UD_\mu U^{-1}. \tag{3.41}$$

Then, if we demand for $\vec{\mathbf{A}}_\mu$ to transform as

$$\vec{\mathbf{A}}'_\mu = U\vec{\mathbf{A}}_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}, \tag{3.42}$$

the covariant derivative (3.39) satisfies the expected transformation property,

$$\begin{aligned}(D_\mu\psi)' &= (\partial_\mu - ig\vec{\mathbf{A}}'_\mu)U\psi \\ &= (\partial_\mu U)\psi + U\partial_\mu\psi - ig\left[U\vec{\mathbf{A}}_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}\right]U\psi \\ &= U(\partial_\mu\psi - ig\vec{\mathbf{A}}_\mu\psi) \\ &= U(D_\mu\psi). \end{aligned} \tag{3.43}$$

For an infinitesimal transformation,

$$\begin{aligned} U &= 1 - i\theta + \mathcal{O}(\theta^2), \\ U^{-1} &= 1 + i\theta + \mathcal{O}(\theta^2), \end{aligned} \tag{3.44}$$

where $\theta = \sum_{i=1}^{3} \frac{\tau^i}{2}\theta^i = \frac{\vec{\tau}}{2}\cdot\vec{\theta}$, (3.42) results in

$$\vec{\mathbf{A}}'_\mu = \vec{\mathbf{A}}_\mu - i\left[\theta, \vec{\mathbf{A}}_\mu\right] - \frac{1}{g}\partial_\mu\theta. \tag{3.45}$$

Using (3.38), we obtain

$$A^{i\prime}_\mu = A^i_\mu + \varepsilon_{ijk}\theta^j A^k_\mu - \frac{1}{g}\partial_\mu\theta^i, \tag{3.46}$$

or

$$\delta A^i_\mu = \varepsilon_{ijk}\theta^j A^k_\mu - \frac{1}{g}\partial_\mu\theta^i. \tag{3.47}$$

To get the gauge field-strength tensor like the electromagnetic field-strength tensor $F_{\mu\nu}$ of (3.33) and its transformation rule under an infinitesimal transformation (3.44), we generalize (3.34) as

$$[D_\mu, D_\nu]\psi = -ig\vec{\mathbf{F}}_{\mu\nu}\psi, \tag{3.48}$$

where $\vec{\mathbf{F}}_{\mu\nu} = \sum_{i=1}^{3}\frac{\tau^i}{2}F^i_{\mu\nu} = \frac{\vec{\tau}}{2}\cdot\vec{F}_{\mu\nu}$ with the gauge field-strength tensor $F^i_{\mu\nu}$ defined by

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\varepsilon_{ijk}A^j_\mu A^k_\nu. \tag{3.49}$$

From the relation

$$([D_\mu, D_\nu]\psi)' = U([D_\mu, D_\nu]\psi), \tag{3.50}$$

which follows from (3.40), one can easily derive

$$\vec{\mathbf{F}}'_{\mu\nu}\psi' = U\vec{\mathbf{F}}_{\mu\nu}\psi \tag{3.51}$$

or

$$\vec{\mathbf{F}}'_{\mu\nu} = U\vec{\mathbf{F}}_{\mu\nu}U^{-1}, \tag{3.52}$$

which leads to the following transformation relation of the field-strength tensor under an infinitesimal transformation (3.44),

$$\delta F^i_{\mu\nu} = \varepsilon_{ijk}\theta^j F^k_{\mu\nu}. \tag{3.53}$$

Note that $F^i_{\mu\nu}$ transforms nontrivially unlike the case of QED where $F_{\mu\nu}$ is invariant because of no structure constant in an Abelian group. In analogy of QED, we construct the kinematic term for the gauge field A^i_μ as

$$-\frac{1}{2}\mathrm{Tr}(\vec{\mathbf{F}}_{\mu\nu}\cdot\vec{\mathbf{F}}^{\mu\nu}) = -\frac{1}{2}\sum_{i,j=1}^{3}\mathrm{Tr}(\frac{\tau^i}{2}F^i_{\mu\nu}\frac{\tau^j}{2}F^{j\mu\nu}) = -\frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} \tag{3.54}$$

where the 2nd equality is due to the relation $\mathrm{Tr}(\frac{\tau^i}{2}\frac{\tau^j}{2}) = \frac{1}{2}\delta^{ij}$. One can easily see that this term is invariant under the gauge transformation as shown in the following;

$$\begin{aligned}\delta(F^i_{\mu\nu}F^{i\mu\nu}) &= 2\delta F^i_{\mu\nu}F^{i\mu\nu}\\ &= 2\varepsilon_{ijk}\theta^j F^k_{\mu\nu}F^{i\mu\nu} = 0,\end{aligned} \tag{3.55}$$

where the last equality is due to the totally anti-symmetric property of ε_{ijk}. Therefore, we can adopt (3.54) as the kinetic term for the gauge field A^i_μ.

In summary, we can write down the gauge invariant Lagrangian for a fermion with mass m in the $SU(2)$ symmetric world as

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_F + \mathcal{L}_G, \qquad (3.56)\\ \mathcal{L}_F &= \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x),\\ \mathcal{L}_G &= -\frac{1}{2}\mathrm{Tr}(\vec{\mathbf{F}}_{\mu\nu}\cdot\vec{\mathbf{F}}^{\mu\nu}) = -\frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu},\end{aligned}$$

where $\mathcal{L}_F$ and $\mathcal{L}_G$ are represent the Lagrangian for the fermion and the kinetic term of gauge fields, respectively. The covariant derivative is defined by (3.39) with 3 gauge fields A^i_μ ($i = 1,2,3$). Transformation property of the gauge field A^i_μ is given by (3.46) or (3.47). The gauge field tensor $F^i_{\mu\nu}$ is defined by (3.49) and its transformation rule under $SU(2)$ gauge transformation is given by (3.52). Change under an infinitesimal transformation (3.44) is given by (3.53). In (3.56), the mass term of gauge fields does not appear again because it violates the gauge symmetry. Some comments are in order for (3.56). (1) Contrary to QED, the kinetic term of gauge fields $\mathcal{L}_G$ contains 3 gauge boson interactions with the same coupling constant g as the one of the gauge field to fermions. It contains also a 4 gauge boson interaction with the coupling g^2 as follows;

$$\mathcal{L}_G = -\frac{1}{2}\partial_\mu A^i_\nu(\partial^\mu A^{i\nu} - \partial^\nu A^{i\mu}) - g\varepsilon_{ijk}A^i_\mu A^j_\nu\partial^\mu A^{k\nu} - \frac{g^2}{4}\varepsilon_{ijk}\varepsilon_{i\ell m}A^j_\mu A^k_\nu A^{\ell\mu}A^{m\nu}. \tag{3.57}$$

(2) The Yang–Mills theory, i.e. the non-Abelian gauge theory is not the free theory even without matter fields because it contains self-interactions among gauge fields. This is very different from the Abelian gauge theory like QED, where there is no

self-coupling of photons. (3) As long as we demand the gauge invariance alone, we can add renormalizable Yukawa interactions such as,

$$\mathcal{L}_Y = G_Y \bar{\psi}(x)\psi(x)\phi(x) \tag{3.58}$$

in which the scalar field $\phi(x)$ is introduced for $\mathcal{L}_Y$ to be invariant under the gauge transformation. There is no principle to determine the form of the Yukawa term except for the requirement of gauge invariance. G_Y is also totally unconstrained; there is no other way to determine it from experiments.

Unfortunately, this theory (3.56) is not useful for weak interaction because it gives identical coupling to right- and left-handed fermions and leads to the parity conservation. To preserve gauge invariance, it is essential to have massless gauge bosons A^i_μ, which should give weak interactions of infinite range like electromagnetic case, being contrary to the real physics. Only if the gauge symmetry is broken by the inclusion of mass term, it becomes possible to achieve agreement with experiment.

3.3 Spontaneous Symmetry Breaking and Goldstone Bosons

Nature seems to possess various types of symmetries such as geometrical (Lorentz invariance, parity and time reversal invariance etc.) and internal (isospin, flavor and color, etc.) symmetries, discrete and continuous symmetries and so on. Some of them are exact symmetries and others are approximate symmetries. Here we are concentrated on the symmetry and its breaking in the physical world. In the field theory, dynamics of the physical world is described by the Lagrangian. There are two ways to discuss the symmetry breaking in the field theory; (1)One way is to add a symmetry breaking term to the symmetric term by hand as

$$\mathcal{L} = \mathcal{L}_{sym} + \mathcal{L}_{breaking}. \tag{3.59}$$

Examples are seen, for example, for the case of an approximate symmetry such as the isospin $SU(2)$ or flavor $SU(3)$ symmetry and so on. This type of symmetry breaking is useful when the symmetry breaking term is small and the perturbative treatment is meaningful. In this case, $\mathcal{L}$ recovers the exact symmetry for vanishing $\mathcal{L}_{breaking}$. However, this case is rather artificial because there is no fundamental principle to determine the exact form of $\mathcal{L}_{breaking}$. (2)Another way is the one called a hidden or a spontaneous symmetry breaking(SSB), where the Lagrangian remains symmetric under certain group transformation while the physical vacuum is made non-invariant. It is well-known that there are many examples of the SSB both in classical and quantum physics such as bent rods under a strong force, infinite feromagnets, crystal lattices, superconductors, etc. Here we are interested in the field theory model of the SSB.

- **Goldstone theorem**

Let us consider the $U(1)$ global invariant Lagrangian composed of a complex scalar field $\phi(x)$ and $\phi^*(x)$,

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi^* \phi), \quad V(\phi^*\phi) = m\phi^*\phi + \lambda(\phi^*\phi)^2. \tag{3.60}$$

The system is equivalent to the one described by the following Lagrangian composed of 2 real fields φ_1 and φ_2 which are related to ϕ and ϕ^* as $\phi = (\varphi_1 + i\varphi_2)/\sqrt{2}$ and $\phi^* = (\varphi_1 - i\varphi_2)/\sqrt{2}$,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi_1 \partial^\mu \varphi_1 + \frac{1}{2}\partial_\mu \varphi_2 \partial^\mu \varphi_2 - V(\varphi_1^2 + \varphi_2^2). \tag{3.61}$$

(3.61) has the $O(2)$ symmetry, that is, the Lagrangian of (3.61) is invariant under the following $O(2)$ transformation,

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{3.62}$$

and one can see that the $U(1)$ symmetry is equivalent to the $O(2)$ symmetry. The potential $V(\phi^*\phi) = V(\varphi_1^2 + \varphi_2^2)$ must have the following properties for the theory to be meaningful; V is (1) at most the 4th order of the fields in order to ensure renormalizability of the theory and (2) bounded below for change of $(\phi^*\phi)^{1/2} = |\phi|$, so that the theory has a stable ground state. A typical example of V is given in (3.60) with $\lambda > 0$.

In quantum field theories, particle excitations of a field are defined as quantized fluctuations of the field about its lowest energy state, i.e. the vacuum state. The constant value of the field corresponding to the lowest energy state is called the vacuum expectation value (VEV), i.e. $\langle 0|\phi|0\rangle \equiv \phi_0$. To find the particle spectra, we expand the potential about its minimum corresponding to the lowest energy state as

$$\begin{aligned} V(\varphi_1, \varphi_2) &= V(\varphi_{01}, \varphi_{02}) + \sum_{a=1,2} \left(\frac{\partial V}{\partial \varphi_a}\right)_0 (\varphi_a - \varphi_{0a}) \\ &+ \frac{1}{2} \sum_{a,b=1,2} \left(\frac{\partial^2 V}{\partial \varphi_a \partial \varphi_b}\right)_0 (\varphi_a - \varphi_{0a})(\varphi_b - \varphi_{0b}) + \cdots, \end{aligned} \tag{3.63}$$

where $\phi_0 = (\varphi_{01}, \varphi_{02})$ is the VEV of $\phi = (\varphi_1, \varphi_2)$, i.e. $\varphi_{0a} = \langle 0|\varphi_a|0\rangle$ $(a = 1, 2)$. Since the potential V has its minimum at $\phi = \phi_0$, the 2nd term of the r.h.s. of (3.63) is zero. The factor $\left(\frac{\partial^2 V}{\partial \varphi_a \partial \varphi_b}\right)_0 \equiv m_{ab}^2$ in the 3rd term is called the mass matrix which is diagonalized to generate the particle spectrum.

Now we have 2 possible cases of V. (1) One is the case where the vacuum is unique and is called "Wigner phase". In this case, only one vacuum state $(\varphi_{01} = 0, \varphi_{02} = 0)$ is realized as shown in Fig. 3.2(a). Let us take the parameters m^2 and

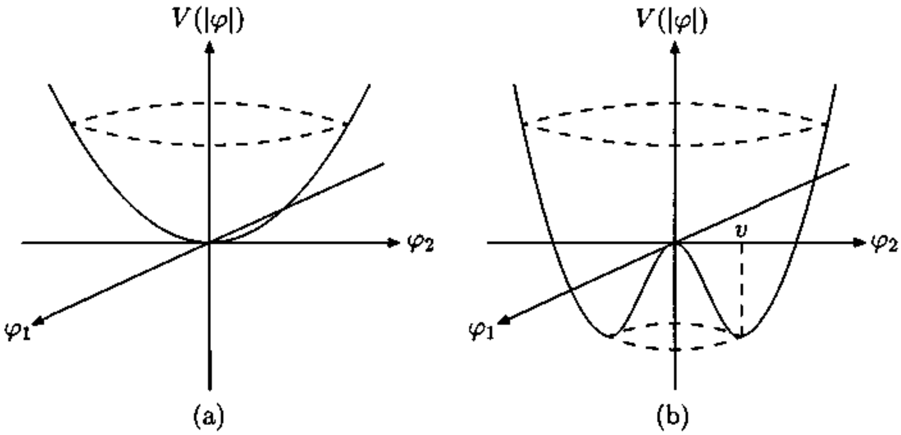


Fig. 3.2

λ for the potential in (3.60) to be positive, $m^2 > 0$ and $\lambda > 0$,

$$V(\varphi_1^2 + \varphi_2^2) = \frac{m^2}{2}(\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2, \tag{3.64}$$

and require the following condition for the vacuum,

$$\begin{aligned} \left(\frac{\partial V}{\partial \varphi_1}\right)_0 &= m^2\varphi_{01} + \lambda\varphi_{01}(\varphi_{01}^2 + \varphi_{02}^2) = 0, \\ \left(\frac{\partial V}{\partial \varphi_2}\right)_0 &= m^2\varphi_{02} + \lambda\varphi_{02}(\varphi_{01}^2 + \varphi_{02}^2) = 0. \end{aligned} \tag{3.65}$$

Then, we are led to the unique vacuum $\varphi_{01} = \varphi_{02} = 0$. The mass matrix becomes diagonal in this case

$$m_{ab}^2 = \begin{pmatrix} m^2 & 0 \\ 0 & m^2 \end{pmatrix}, \tag{3.66}$$

which means that φ_1 and φ_2 have the same mass m as already seen from (3.64). (2) Another one is the case where the vacuum is not unique and called "Nambu–Goldstone phase". This case is realized, for example, for continuously or infinitely degenerated vacuum states with $\varphi_{01} \neq 0$ and/or $\varphi_{02} \neq 0$, as shown in Fig. 3.2(b), for the potential with $m^2 = -\mu^2$ ($\mu^2 > 0$) and $\lambda > 0$,

$$V(\varphi_1^2 + \varphi_2^2) = -\frac{\mu^2}{2}(\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2. \tag{3.67}$$

The minimum of V is realized by requiring

$$\left(\frac{\partial V}{\partial \varphi_1}\right)_0 = -\mu^2\varphi_{01} + \lambda\varphi_{01}(\varphi_{01}^2 + \varphi_{02}^2) = 0,$$

$$\left(\frac{\partial V}{\partial \varphi_2}\right)_0 = -\mu^2\varphi_{02} + \lambda\varphi_{02}(\varphi_{01}^2 + \varphi_{02}^2) = 0, \tag{3.68}$$

which leads to the condition

$$\varphi_{01}^2 + \varphi_{02}^2 \equiv v^2 = \frac{\mu^2}{\lambda} \quad \text{or} \quad (\phi^*\phi)_0 \equiv |\phi_0|^2 = \frac{v^2}{2} = \frac{\mu^2}{2\lambda}. \tag{3.69}$$

In other words, all points on a circle with radius $v = \sqrt{\mu^2/\lambda}$ in the (φ_1, φ_2) plane correspond to the minimum of V, that is, the vacuum state is no longer unique but is $O(2)$ symmetric. One can choose any point as the physical vacuum. From (3.67), we obtain

$$\begin{aligned}
\frac{\partial^2 V}{\partial \varphi_1^2} &= \left(-\mu^2 + \lambda(\varphi_1^2 + \varphi_2^2)\right) + 2\lambda\varphi_1^2, \\
\frac{\partial^2 V}{\partial \varphi_2^2} &= \left(-\mu^2 + \lambda(\varphi_1^2 + \varphi_2^2)\right) + 2\lambda\varphi_2^2, \\
\frac{\partial^2 V}{\partial \varphi_1 \partial \varphi_2} &= 2\lambda\varphi_1\varphi_2.
\end{aligned} \tag{3.70}$$

Then, if we choose a point $(\varphi_{01} = v, \varphi_{02} = 0)$ as the physical vacuum, we obtain the mass matrix as

$$m_{ab}^2 = \begin{pmatrix} 2\lambda v^2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.71}$$

Therefore, we find that $\varphi_1' = \varphi_1 - v$ corresponds to a massive particle with mass $m^2 = 2\lambda v^2$, while $\varphi_2' = \varphi_2$ is massless. φ_2' is called a "Goldstone boson". Actually using these new fields, we can rewrite the Lagrangian (3.61) as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi_1')^2 + \frac{1}{2}(\partial_\mu\varphi_2')^2 - \frac{1}{2}(2\lambda v^2)\varphi_1'^2 + \lambda v\varphi_1'(\varphi_1'^2 + \varphi_2'^2) - \frac{\lambda}{4}(\varphi_1'^2 + \varphi_2'^2)^2. \tag{3.72}$$

This Lagrangian (3.72) has no longer $O(2)$ symmetry, though the original Lagrangian (3.61) has it explicitly. That is to say, the symmetry of the original Lagrangian has been broken by breaking the symmetry of vacuum. This is called a hidden symmetry or spontaneous symmetry breaking(SSB).

In summary, starting from the Lagrangian which has a global symmetry, taking the negative parameter $m^2 = -\mu^2$ $(\mu^2 > 0)$, and breaking the symmetry of the vacuum states by choosing a particular point among symmetrically degenerate vacuum states, we found that a massless particle, called the Goldstone boson, appeared. This mechanism is called the Goldstone theorem. (Goldstone, 1961; Goldstone, Salam and Weinberg, 1962; Bludman and Klein, 1962) In general, when a global symmetry is spontaneously broken, a number of massless Goldstone bosons appear depending on the symmetry properties.

- **Useful parametrization for the $U(1)$ model**

In literature, another and more useful parametrization of the field is often used. Introducing the two real fields $\rho(x)$ and $\theta(x)$, we can write the complex field $\phi(x)$ in (3.60) as

$$\phi(x) = \frac{1}{\sqrt{2}}\rho(x)e^{i\theta(x)/v}, \tag{3.73}$$

where v is a constant given by (3.69). Then, substituting

$$\partial_\mu\phi = \frac{1}{\sqrt{2}}e^{i\theta/v}(\partial_\mu\rho + \frac{i}{v}\rho\partial_\mu\theta), \tag{3.74}$$

into (3.60), we have

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\rho)^2 + \frac{1}{2v^2}\rho^2(\partial_\mu\theta)^2 - V(\rho^2). \tag{3.75}$$

To find the mass of the particle corresponding to the excitation of the would-be radial field $\rho(x)$, we expand it as $\rho(x) = v + \eta(x)$ and obtain

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\eta)^2 + \frac{1}{2}(\partial_\mu\theta)^2 + \frac{1}{v}\eta(\partial_\mu\theta)^2 + \frac{1}{2v^2}\eta^2(\partial_\mu\theta)^2 - V(\rho^2), \tag{3.76}$$

with

$$V(\rho^2) = \frac{1}{2}(2\mu^2)\eta^2 + \lambda v\eta^3 + \frac{\lambda}{4}\eta^4 - \frac{1}{4}\mu^2v^2. \tag{3.77}$$

It is noted that there is no quadratic term of θ in this Lagrangian. From this Lagrangian, one can easily find that we have a massive η field with mass $m_\eta = \sqrt{2\mu^2}$ and a massless Goldstone boson θ.

- **Extention to the $SU(2)$ model**

In the $SU(2)$ model, the field is given as a doublet, $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, composed of 2 complex fields, $\phi_1 = (\varphi_1 + i\varphi_2)/\sqrt{2}$ and $\phi_2 = (\varphi_3 + i\varphi_4)/\sqrt{2}$. Then, the global $SU(2)$ invariant Lagrangian is given as

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - V(\phi^\dagger\phi), \tag{3.78}$$

$$V = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2, \tag{3.79}$$

with $\mu^2 > 0$ as before. Now, by using 4 real fields $H(x)$ and $\xi^i(x)$ $(i = 1, 2, 3)$, we write ϕ as

$$\phi = \frac{1}{\sqrt{2}}e^{i\tau^i\xi^i(x)/2v}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}, \tag{3.80}$$

where the VEV v is defined as $\varphi_{01}^2+\varphi_{02}^2+\varphi_{03}^2+\varphi_{04}^2 \equiv v^2 = \frac{\mu^2}{\lambda}$ or $(\phi^\dagger\phi)_0 \equiv |\phi_0|^2 = \frac{v^2}{2} = \frac{\mu^2}{2\lambda}$. Then, putting

$$\partial_\mu\phi = \frac{1}{\sqrt{2}}e^{i\tau^i\xi^i/2v}\left\{\begin{pmatrix} 0 \\ \partial_\mu H\end{pmatrix} + \frac{i}{v}\frac{\tau^i}{2}\partial_\mu\xi^i\begin{pmatrix} 0 \\ v+H\end{pmatrix}\right\}, \tag{3.81}$$

into (3.78), we get

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left\{(0 \;\; \partial_\mu H) - \frac{i}{v}\partial_\mu\xi^i(0 \;\; v+H)\frac{\tau^i}{2}\right\} \\ &\times \left\{\begin{pmatrix} 0 \\ \partial^\mu H\end{pmatrix} + \frac{i}{v}\frac{\tau^j}{2}\partial^\mu\xi^j\begin{pmatrix} 0 \\ v+H\end{pmatrix}\right\} - V\left((v+H)^2\right),\end{aligned} \tag{3.82}$$

which leads to

$$\mathcal{L} = \frac{1}{2}\partial_\mu H\partial^\mu H + \frac{1}{8v^2}\partial_\mu\xi^i\partial^\mu\xi^i(v+H)^2 - V\left((v+H)^2\right), \tag{3.83}$$

using the relation $\tau^i\tau^j = \delta^{ij} + i\varepsilon_{ijk}\tau^k$. From this Lagrangian, one can see that the 3 fields ξ^i $(i=1,2,3)$ have no mass terms. They are massless Goldstone bosons, while H is massive with mass $m_H = \sqrt{2\mu^2}$. The original $SU(2)$ symmetry disappeared. It is remarkable to note that the number of Goldstone bosons is equal to the number of generators breaking the symmetry of the vacuum state. In an example of $SU(2)$ model, the number of Goldstone bosons are three, ξ^i $(i=1,2,3)$, corresponding to the 3 generators, τ^i $(i=1,2,3)$.

This is generalized for cases with larger symmetries. Now, let us consider an $SU(n)$ symmetric world with n component fields,

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \cdot \\ \cdot \\ \cdot \\ \varphi_n \end{pmatrix}. \tag{3.84}$$

The $SU(n)$ group has $N = n^2-1$ generators T^i $(i=1,2,\cdots,N)$ and under this group, ϕ transforms as follows;

$$\phi \to \phi' = e^{-i\theta^i T^i}\phi, \tag{3.85}$$

or under an infinitesimal transformation,

$$\delta\phi = -i\theta^i T^i\phi, \tag{3.86}$$

where the summation over i is implied. T^i are the matrices with $n\times n$ components. Suppose that the vacuum is not unique but is symmetrically degenerate. Then, choose a particular vacuum state,

$$\phi_0 = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \cdot \\ \cdot \\ \cdot \\ \varphi_n \end{pmatrix}_0 . \tag{3.87}$$

Under an infinitesimal transformation, a change of this vacuum state is $\delta\phi_0 = -i\theta^i T^i \phi_0$. Then, one can see that the vacuum is not invariant under operation of generators T^i if $T^i\phi_0 \neq 0$. Suppose that $M(\leq N)$ generators T^i $(i = 1, 2, \cdots, M)$ are such generators, while other T^i $(i = M+1, M+2, \cdots, N)$ leave the vacuum invariant as $T^i\phi_0 = 0$. Then, by introducing a new field $\phi' = \phi - \phi_0$, we expand the potential V around the vacuum as before,

$$V = V_0 + \sum_{a=1}^{n} \left(\frac{\partial V}{\partial \varphi_a}\right)_0 \varphi'_a + \frac{1}{2}\sum_{a,b=1}^{n} \left(\frac{\partial^2 V}{\partial \varphi_a \partial \varphi_b}\right)_0 \varphi'_a \varphi'_b + \cdots, \tag{3.88}$$

with

$$\left(\frac{\partial V}{\partial \varphi_a}\right)_0 = 0, \quad \text{and} \quad m^2_{ab} = \left(\frac{\partial^2 V}{\partial \varphi_a \partial \varphi_b}\right)_0 . \tag{3.89}$$

The potential should be invariant under $SU(n)$, i.e. $\delta V = \sum_a \frac{\partial V}{\partial \varphi_a}\delta\varphi_a = 0$ which, by using (3.86), results in

$$\sum_{a,b} \frac{\partial V}{\partial \varphi_a} T^i_{ab}\varphi_b = 0, \quad \text{for} \quad i = 1, 2, \cdots, N. \tag{3.90}$$

Then, by differentiating this equation by φ_c, we obtain

$$\sum_{a,b} \frac{\partial^2 V}{\partial \varphi_c \partial \varphi_a} T^i_{ab}\varphi_b + \sum_a \frac{\partial V}{\partial \varphi_a} T^i_{ac} = 0, \tag{3.91}$$

which leads to, at the vacuum $\phi = \phi_0$,

$$\sum_{a,b} m^2_{ca} T^i_{ab} (\varphi_b)_0 = 0, \quad \text{for} \quad i = 1, 2, \cdots, N. \tag{3.92}$$

As described above, $T^i\varphi_0 \neq 0$ for $i = 1, 2, \cdots, M$. Therefore, $T^i\varphi_0$ span a M-dimensional subspace in the n dimensional space spanned by ϕ of (3.84) and thus, m^2_{ca} has M zero eigenvalues. In other words, there exist M Goldstone bosons or to each generator which breaks the symmetry, a Goldstone boson appears.

For more detailed discussion on the spontaneous symmetry breaking and Goldstone bosons, see, for example, reviews by Abers and Lee (Abers and Lee, 1973) or Coleman (Coleman. 1975).

3.4 Higgs Mechanism

In the previous section, we discussed the spontaneous symmetry breaking under the global symmetry, i.e. the Goldstone theorem. Here we go further by extending the global symmetry to the local gauge symmetry.

• **The $U(1)$ model**

First, let us consider an example of the scalar electrodynamics for a scalar complex field $\phi = \varphi_1 + i\varphi_2$, which is local $U(1)$ invariant. The Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\phi^*\phi), \tag{3.93}$$

$$V(\phi^*\phi) = -\mu^2\phi^*\phi + \lambda(\phi^*\phi)^2, \tag{3.94}$$

with $\mu^2 > 0$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field-strength tensor for photon fields A_μ and $D_\mu\phi$ is the covariant derivative $D_\mu\phi = (\partial_\mu - ieA_\mu)\phi$ as before. The Lagrangian (3.93) is invariant under the following $U(1)$ gauge transformation,

$$\phi \to \phi' = e^{-i\alpha(x)}\phi, \tag{3.95}$$

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\alpha(x). \tag{3.96}$$

Now let us minimize the potential V as before with $|\phi_0|^2 \equiv \frac{v^2}{2} = \frac{\mu^2}{2\lambda}$. Then, if we parametrize the field $\phi(x)$ as

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x))e^{i\theta(x)/v}, \tag{3.97}$$

where $\eta(x)$ and $\theta(x)$ are real fields. We can define a new set of fields by taking a particular gauge transformation with $\alpha(x) = \theta(x)/v$, which is called the "unitary gauge",

$$\phi(x) \to \phi(x)' = e^{-i\theta(x)/v}\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x)), \tag{3.98}$$

$$A_\mu(x) \to B_\mu(x) = A_\mu(x) - \frac{1}{ev}\partial_\mu\theta(x). \tag{3.99}$$

Under this unitary gauge transformation, we have

$$D_\mu\phi(x) \to D'_\mu\phi'(x) = (\partial_\mu - ieB_\mu)\frac{1}{\sqrt{2}}(v + \eta(x))) \tag{3.100}$$

and

$$F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu \to F_{\mu\nu}(B) = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{3.101}$$

Here we can easily show $F_{\mu\nu}(B) = F_{\mu\nu}(A)$ by substituting (3.99) into (3.101), that is, the field–strength tensor $F_{\mu\nu}$ is gauge invariant as it should be. Substituting (3.98) $\sim$ (3.101) into (3.93), one can rewrite the Lagrangian as follows;

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}|\partial_\mu \eta - ieB_\mu(v+\eta)|^2 - \frac{\mu^2}{2}(v+\eta)^2 - \frac{\lambda}{4}(v+\eta)^4 - \frac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B) \\
&= \frac{1}{2}\partial_\mu \eta \partial^\mu \eta - \mu^2\eta^2 - \frac{1}{4}F_{\mu\nu}(B)F^{\mu\nu}(B) + \frac{1}{2}(ev)^2 B_\mu B^\mu \\
&+ \frac{1}{2}e^2 B_\mu B^\mu \eta(\eta + 2v) - \lambda v \eta^3 - \frac{\lambda}{4}\eta^4. \qquad (3.102)
\end{aligned}
$$

We can see that this Lagrangian describes a massive vector boson B with mass $m_B = ev$ and a massive scalar η with mass $m_\eta = \sqrt{2\mu^2}$. η is called "Higgs boson". Here we have no Goldstone boson which has gone out of the Lagrangian (3.102).

In summary, by extending the symmetry of the Lagrangian from the global to local one, we found that the massless Goldstone boson θ disappeared and a massive gauge vector boson B and a massive scalar boson η called Higgs boson came out. This is called the Higgs mechanism. (Higgs, 1964, 1966; Englert and Brout, 1964; Guralnik, Hagen and Kibble, 1965; Kibble 1967) The Goldstone boson θ was eaten up by the gauge boson B and became the longitudinal component of it. It is remarkable to note that in the Higgs mechanism the degree of freedom is conserved, that is to say, starting from 2 real scalar fields (φ_1, φ_2) or (η, θ) plus 2 polarization states of massless photons A_μ, we finally got one real massive scalar field η and one massive vector boson B_μ with 3 polarization degrees of freedom

- **The $SU(2)$ model**

Let us next consider the non-Abelian $SU(2)$ model by generalizing the $U(1)$ model. We have again the complex doublet field $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. Then, the gauge invariant Lagrangian with $SU(2)$ symmetry is given by

$$
\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) - \frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} - V(\phi^\dagger\phi), \qquad (3.103)
$$

with

$$
\begin{aligned}
D_\mu\phi &= (\partial_\mu - ig\frac{\tau^i}{2}A^i_\mu)\phi, \quad (i = 1, 2, 3) \qquad (3.104) \\
F^i_{\mu\nu} &= \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\varepsilon_{ijk}A^j_\mu A^k_\nu, \qquad (3.105) \\
V(\phi^\dagger\phi) &= -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2. \quad (\mu^2 > 0) \qquad (3.106)
\end{aligned}
$$

Now, introducing the new real fields $H(x)$ and $\xi^i(x)$ $(i = 1, 2, 3)$, let us parametrize the field $\phi(x)$ as

$$
\phi(x) = \frac{1}{\sqrt{2}}e^{i\tau^i\xi^i(x)/2v}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}, \qquad (3.107)
$$

Then, by taking the unitary gauge as before, we define the new fields as

$$\phi(x) \quad \to \quad \phi'(x) = U(x)\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}, \tag{3.108}$$

$$\vec{\mathbf{A}}_\mu \quad \to \quad \vec{\mathbf{B}}_\mu = U(x)\vec{\mathbf{A}}_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}, \tag{3.109}$$

with

$$U(x) = e^{-i\tau^i\xi^i(x)/2v}, \tag{3.110}$$

where the summation over i is implied. This transformation leads to

$$D_\mu\phi \quad \to \quad (D_\mu\phi)' = (\partial_\mu - ig\frac{\tau^i}{2}B^i_\mu)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}, \tag{3.111}$$

$$F^i_{\mu\nu}(A)F^{i\mu\nu}(A) \quad \to \quad F^i_{\mu\nu}(B)F^{i\mu\nu}(B) = F^i_{\mu\nu}(A)F^{i\mu\nu}(A), \tag{3.112}$$

with

$$F^i_{\mu\nu}(B) = \partial_\mu B^i_\nu - \partial_\nu B^i_\mu + g\varepsilon_{ijk}B^j_\mu B^k_\nu. \tag{3.113}$$

Then the Lagrangian becomes

$$\mathcal{L} = (D_\mu\phi)'^\dagger(D^\mu\phi)' - \frac{1}{4}F^i_{\mu\nu}(B)F^{i\mu\nu}(B) + \mu^2(\phi'^\dagger\phi') - \lambda(\phi'^\dagger\phi')^2. \tag{3.114}$$

From this Lagrangian we can see that the three $\xi^i(x)$ $(i = 1,2,3)$ fields disappeared. Where did these fields go? We can find the answer by writing $\mathcal{L}$ in terms of the component fields of ϕ'. First, let us write down the covariant derivative term,

$$\begin{aligned} [(D_\mu\phi)']^{\dagger a}(D^\mu\phi)'_a &= \frac{1}{2}\partial_\mu H\partial^\mu H + g^2 B^i_\mu B^{j\mu}\left(\frac{\tau^i}{2}\right)^a_b\left(\frac{\tau^j}{2}\right)^c_a\phi'^b\phi'_c, \\ &= \frac{1}{2}\partial_\mu H\partial^\mu H + \frac{g^2}{8}B^i_\mu B^{i\mu}(v+H)^2. \end{aligned} \tag{3.115}$$

Then, we finally obtain the following Lagrangian,

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu H\partial^\mu H - \mu^2H^2 - \frac{1}{4}F^i_{\mu\nu}(B)F^{i\mu\nu}(B) + \frac{g^2v^2}{8}B^i_\mu B^{i\mu} \\ &+ \frac{g^2}{8}B^i_\mu B^{i\mu}H(2v+H) - \lambda vH^3 - \frac{\lambda}{4}H^4 - \frac{v^4}{4}. \end{aligned} \tag{3.116}$$

A triplet of massive vector fields B^i_μ $(i = 1,2,3)$ with mass $m_B = \frac{1}{2}gv$ and a single massive scalar, i.e. Higgs boson H with mass $m_H = \sqrt{2\mu^2}$ appeared. We found here again that the Goldstone bosons ξ^i $(i = 1,2,3)$ were eaten by the gauge bosons B^i $(i = 1,2,3)$ to make their longitudinal components. This is the Higgs mechanism in the non-Abelian $SU(2)$ gauge theory. It is again interesting to note that the number of degrees of freedom is conserved in the Higgs mechanism, i.e. the 3 Goldstone bosons ξ^i $(i = 1,2,3)$ become the longitudinal components of the respective gauge fields which lead to appearance of 3 massive vector bosons B^i $(i = 1,2,3)$. The discovery of the spontaneous symmetry breaking (SSB) and the Higgs mechanism in the non-Abelian gauge theories made a great breakthrough toward the unification of electromagnetic and weak interactions, which will be discussed in detail in the next Chapter.

Problems

3.1 Show that when the generators T^i satisfy the Lie algebra (3.17),

$$[T^i, T^j] = if_{ijk}T^k,$$

the "charge" Q^i also satisfy the same Lie algebra (3.16),

$$[Q^i, Q^j] = if_{ijk}Q^k,$$

where Q^i is defined by (3.13).

3.2 Using the canonical commutation relations of (3.4) and (3.5), prove the relation (3.18).

3.3 In the SSB of the global $O(2)$ symmetric theory described in 3.3, discuss what happens if we take the vacuum to be ($\varphi_{01} = \frac{v}{\sqrt{2}}$, $\varphi_{02} = \frac{v}{\sqrt{2}}$) instead of ($\varphi_{01} = v$, $\varphi_{02} = 0$), and show that both cases are physically equivalent.

3.4 Derive (3.116).

3.5 Consider the global $O(n)$ symmetric Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^T(x) \cdot \partial^\mu \phi - V(\phi^T \cdot \phi), \quad \text{with} \quad V(\phi^T \cdot \phi) = \frac{m^2}{2}\phi^T \cdot \phi + \frac{\lambda}{4}(\phi^T \cdot \phi)^2,$$

where

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{pmatrix}.$$

After the spontaneous breaking of this $O(n)$ symmetry by taking $m^2 = -\mu^2$ ($\mu^2 > 0$), show that one can make the vacuum to be invariant under $O(n-1)$ transformation, that is, $O(n)$ breaks into $O(n-1)$. Furthermore, show that the number of Goldstone bosons is equal to the number of broken generators of the original $O(n)$ group.

Chapter 4

THE STANDARD MODEL OF ELECTROWEAK INTERACTIONS

In this Chapter, we describe the Glashow–Weinberg–Salam(GWS) model of electroweak interactions (Glashow, 1961; Weinberg, 1967; Salam, 1968). It is a non-Abelian gauge theory with $SU(2)_L \times U(1)_Y$ gauge symmetry accompanied by the Higgs mechanism. It is the first successful model toward the unified theory of elementary particle interactions. The model is extremely successful in particle physics phenomenology without serious discrepancy with almost all existing data; only one exception at present might be an evidence of massive neutrino established with recent observation of neutrino oscillation. The discovery of $W^{\pm}$ and Z^0 bosons with expected masses and a weak neutral current mediated by a massive neutral vector boson Z^0 is a great triumph of the model. Here we first consider the model for one lepton family of an electron e and the corresponding neutrino ν_e. Then, the realistic case of more lepton families is described. Extension to the quark sector is also discussed, where the non-diagonal quark mass matrices and the Cabibbo–Kobayashi–Maskawa(CKM) matrix are introduced. These embody the standard model of the electroweak interaction.

4.1 Fermions in the GWS Model

Let us start with the discussion on the one family of leptons, an electron e and its neutrino ν_e. ν_e is considered to be massless in this model, while e is massive with a small mass, $m_e \simeq 0.5$MeV. Both of them have spin 1/2. In this section, we assign these particles to the appropriate representations of $SU(2)_L \times U(1)_Y$ gauge symmetry, the gauge symmetry of the GWS model.

As we have seen in chapter 2, in the weak processes such as

$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu, \tag{4.1}$$

$$\pi^- \rightarrow \mu^- + \bar{\nu}_\mu, \tag{4.2}$$

$$n \rightarrow p + e^- + \bar{\nu}_e, \tag{4.3}$$

only left-handed leptons and right-handed anti-leptons take part in and the decay amplitudes of these processes can be written down in terms of charged currents

$$\begin{aligned} J_\mu(x) \equiv J_\mu(x)^+ &= \bar{\nu}_{eL}(x)\gamma_\mu e_L(x) \\ &= \frac{1}{2}\bar{\nu}_e(x)\gamma_\mu(1-\gamma_5)e_(x), \end{aligned} \tag{4.4}$$

$$\begin{aligned} J_\mu(x)^\dagger \equiv J_\mu(x)^- &= \bar{e}_L(x)\gamma_\mu \nu_{eL} \\ &= \frac{1}{2}\bar{e}(x)\gamma_\mu(1-\gamma_5)\nu_e(x). \end{aligned} \tag{4.5}$$

This suggests that e_L and ν_{eL} could be arranged to simply make a doublet associated with $SU(2)$ group. Let us introduce a lepton doublet composed of the left-handed components of fermions,

$$L = \frac{(1-\gamma_5)}{2}\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L . \tag{4.6}$$

Then, by using this doublet and 2×2 matrices in the so-called weak isospin space

$$\tau^+ = \frac{\tau^1 + i\tau^2}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{4.7}$$

$$\tau^- = \frac{\tau^1 - i\tau^2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{4.8}$$

one can rewrite the charged currents (4.4) and (4.5) as follows;

$$J_\mu^+ = \bar{L}\gamma_\mu \tau^+ L, \tag{4.9}$$

$$J_\mu^- = \bar{L}\gamma_\mu \tau^- L. \tag{4.10}$$

(Note that in this Chapter τ^i are used to denote Pauli matrices, in order to make clear that they stand for the weak isospin, not the ordinary spin.) These forms suggest that the weak currents make an $SU(2)$ group by introducing an additional neutral current,

$$\begin{aligned} J_\mu^3 &= \bar{L}\gamma_\mu \frac{\tau^3}{2} L \\ &= \frac{1}{2}\bar{\nu}_e\gamma_\mu\nu_e - \frac{1}{2}\bar{e}_L\gamma_\mu e_L. \end{aligned} \tag{4.11}$$

Then, we have 2 charged and 1 neutral currents, $J_\mu^\pm$ and J_μ^3, which couple to the weak bosons $W_\mu^\pm$ and A_μ^3, respectively, just as the electromagnetic current $J_\mu^{em}(x)$ couples to photon $A_\mu(x)$, as we will see later. Existence of $J_\mu^\pm$ and J_μ^3 suggests that we have weak isotopic triplet currents of $SU(2)$, belonging to the adjoint representation,

$$J_\mu^i(x) = \bar{L}\gamma_\mu T^i L = \bar{L}\gamma_\mu \frac{\tau^i}{2} L, \quad (i = 1, 2, 3). \tag{4.12}$$

It is easily shown that the corresponding charges T^i $(i = 1, 2, 3)$ defined by

$$T^i = \int J_0^i(x) d^3x, \tag{4.13}$$

satisfies the $SU(2)_L$ algebra

$$[T^i, T^j] = i\varepsilon^{ijk} T^k, \tag{4.14}$$

with ε^{ijk} being the totally antisymmetric Levi-Cività tensor with $\varepsilon^{123} = 1$. In (4.12) the subscript L of $SU(2)_L$ means that the weak isospin currents are composed of only the left-handed Weyl fermions. In all of these currents, the right-handed component of electron e_R does not appear. It has no interactions with any other particles and thus it should be a singlet under $SU(2)_L$ transformation, i.e.

$$R = \frac{1}{2}(1 + \gamma_5)e = e_R. \tag{4.15}$$

As mentioned above, in this Chapter we assume ν_e to be massless and thus ν_e has no right-handed component. At a first glance, one might think of identifying the neutral current J_μ^3 of (4.11) with the electromagnetic one, J_μ^{em}. But it is impossible because of the following reasons; (1)the neutral current $J_\mu^3(x)$ has no right-handed component, while the electromagnetic current $J_\mu^{em}(x)$ for an electron has both left-handed and right-handed components as follows, essentially because parity is a good symmetry in the electromagnetic interactions;

$$J_\mu^{em}(x) = -\bar{e}\gamma_\mu e = -\bar{e}_L\gamma_\mu e_L - \bar{e}_R\gamma_\mu e_R, \tag{4.16}$$

(2)the "charge" of neutrino which couples to A_μ^3 is not zero but opposite to the one of electron as shown in (4.11), while it should be zero for the electromagnetic interaction, and (3)contrary to the electromagnetic current, the neutral current J_μ^3 maximally violates parity. These arguments suggest that gauge symmetry $SU(2)_L$ should be enlarged. Let us note that, though (4.11) leads to a wrong relation saying that the sum of electric charges of ν_e and e^- vanish, due to the property Tr $\tau^3 = 0$, it gives the correct difference of the charges, i.e. $\frac{1}{2} - (-\frac{1}{2}) = 1 - 0$, This implies that J_μ^{em} should have a piece generated by a diagonal 2×2 generator with non-vanishing trace. Namely the gauge group should be enlarged so that it contains a new $U(1)$ symmetry. Therefore, we also need another gauge field B_μ associated with the new $U(1)$ symmetry. This new $U(1)$ group should be independent of $SU(2)_L$ group and thus its generator should commute with the generators of $SU(2)_L$, T^i $(i = 1, 2, 3)$. The gauge group is thus extended to the direct product of $SU(2)_L$ and $U(1)$. Then, how can we practically realize it? We want to keep the form of the electromagnetic current for a fermion ψ with charge Q to be

$$J_\mu^{em}(x) = \bar{\psi}\gamma_\mu Q\psi, \tag{4.17}$$

where Q is the charge for the fermion ψ. For the case of an electron, the eigenvalue of Q is $Q = -1$. From (4.17), we can define the generator of $U(1)_{em}$ for an electron,

lepton family	Q	(T, T^3)	Y
ν_e, ν_μ, ν_τ	0	$(\frac{1}{2}, +\frac{1}{2})$	-1
e_L, μ_L, τ_L	-1	$(\frac{1}{2}, -\frac{1}{2})$	-1
e_R, μ_R, τ_R	-1	0	-2

Table 4.1 Quantum number assignment of lepton families in the GWS model.

i.e. the charge of an electron as

$$Q = \int J(x)_0^{em} d^3x = -\int \bar{e}\gamma_0 e d^3x = -\int (e_L^\dagger e_L + e_R^\dagger e_R) d^3x. \tag{4.18}$$

(The reader should not be confused by this notation, though the same Q as in (4.17) is used here.) However, since this generator Q does not commute with T^i defined by (4.13), $U(1)_{em}$ and $SU(2)_L$ cannot be simultaneous symmetries of the model. Then, we look for a new $U(1)$ symmetry, called $U(1)_Y$, so that Q is given by a linear combination of the generator T^3 of the $SU(2)_L$ group and the generator of this new $U(1)_Y$ group. In doing so, it is interesting to note the relation

$$Q - T^3 = \int d^3x \left(-\frac{1}{2}\nu_{eL}^\dagger \nu_{eL} - \frac{1}{2} e_L^\dagger e_L - e_R^\dagger e_R \right), \tag{4.19}$$

which shows that each element in an $SU(2)_L$ doublet $(\nu_e\ e)_L^t$ has the same eigenvalue $-\frac{1}{2}$ and the eigenvalue of e_R is -1. Furthermore, $Q - T^3$ commutes with $T^i (i = 1, 2, 3)$, i.e. $[Q - T^3, T^i] = 0$, that is, $Q - T^3$ and T^i can be simultaneous symmetries of the model. Thus, it is reasonable to define this new generator of the $U(1)_Y$ group as $\frac{Y}{2} = Q - T^3$ or

$$Q = T^3 + \frac{Y}{2}. \tag{4.20}$$

The eigenvalue of Y is called weak hypercharge; the name of this quantum number is originated from the fact that (4.20) is of the same form as Nakano–Nishijima–Gell-Mann relation (Nakano and Nishijima 1953; Gell-Mann 1953), established in 1950s in successfully describing the hadron classification, in which Y was called hypercharge. These quantum numbers for an electron e and its neutrino ν_e are summarized in Table 4.1. (In this table, the quantum numbers of muon and tau families are also given,)

In summary, the GWS model is the $SU(2)_L \times U(1)_Y$ gauge theory and we have, as matter fields, a left-handed doublet L and a right-handed singlet R of $SU(2)$ group which are represented as

$$L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad R = e_R. \tag{4.21}$$

The model starts with the Lagrangian constructed with L and R, which is invariant under the direct product of $SU(2)_L$ and $U(1)_Y$ groups:

$$
\begin{aligned}
SU(2)_L \quad &: \quad L \to L' = e^{-i\alpha^i(x)\frac{\tau^i}{2}} L, \quad R \to R' = R, \\
U(1)_Y \quad &: \quad L \to L' = e^{\frac{i}{2}\beta(x)} L, \quad R \to R' = e^{i\beta(x)} R,
\end{aligned} \tag{4.22}
$$

where α^i $(i = 1, 2, 3)$ and β are group parameters for weak isospin and weak hypercharge operators, respectively. Since we are considering the local gauge invariant Lagrangian, α^i and β are x-dependent.

4.2 $SU(2)_L \times U(1)_Y$ Invariant Lagrangian

In the previous section, we introduced the representation of an electron and its neutrino with respect to the $SU(2)_L \times U(1)_Y$ symmetry, which is given in (4.21). The gauge invariant Lagrangian with $SU(2)_L \times U(1)_Y$ symmetry for these fermions is constructed as

$$
\begin{aligned}
\mathcal{L}_F \quad &= \quad \bar{L} i\gamma^\mu (\partial_\mu - ig\frac{\vec{\tau}}{2} \cdot \vec{A}_\mu + \frac{i}{2} g' B_\mu) L \\
&+ \quad \bar{R} i\gamma^\mu (\partial_\mu + ig' B_\mu) R,
\end{aligned} \tag{4.23}
$$

where $A^i_\mu (i = 1, 2, 3)$ and B_μ are gauge boson fields associated with $SU(2)_L$ and $U(1)_Y$, respectively. g and g' are the gauge coupling constants corresponding to $SU(2)_L$ and $U(1)_Y$, respectively. Here the explicit forms of the covariant derivatives for L and R come out from the general form

$$
D_\mu = \partial_\mu - ig\frac{\vec{\tau}}{2} \cdot \vec{A}_\mu - ig'\frac{Y}{2} B_\mu, \tag{4.24}
$$

by taking account of $Y = -1$ for L and $Y = -2$ for R, respectively (see Table 4.1). R is a singlet of $SU(2)_L$ and hence does not couple to A^i_μ. In (4.23), the fermion mass term, which connects L and R fields, does not appear because it violates $SU(2)_L \times U(1)_Y$ invariance. Therefore, all fermions, i.e. an electron and its neutrino, are massless at this stage.

The kinetic term of the gauge fields which should be added to $\mathcal{L}_F$ is given by

$$
\mathcal{L}_G = -\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \tag{4.25}
$$

with

$$
F^i_{\mu\nu} \quad = \quad \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\varepsilon_{ijk} A^j_\mu A^k_\nu, \tag{4.26}
$$

$$
B_{\mu\nu} \quad = \quad \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{4.27}
$$

where $F^i_{\mu\nu} (i = 1, 2, 3)$ and $B_{\mu\nu}$ are field strength tensors of gauge fields corresponding to $SU(2)_L$ and $U(1)_Y$, respectively. As is well known, the mass terms of these gauge bosons do not appear because of the local gauge invariance.

In order to make fermions and gauge bosons massive and make the real world, we need spontaneous breakdown of gauge invariance, i.e. the Higgs mechanism, described in the previous chapter. Since we are living in the $U(1)_{em}$ symmetric world with massless photon, we need to have the following breaking,

$$SU(2)_L \times U(1)_Y \longrightarrow U(1)_{em}. \tag{4.28}$$

To realize this symmetry breaking, we introduce the scalar fields, called Higgs bosons, which give rise to the Higgs mechanism. Since we start with 4 gauge bosons(3 associated with $SU(2)_L$ and 1 with $U(1)_Y$) and finally want to have 1 massless photon associated with $U(1)_{em}$, we need scalars with at least 4 degrees of freedom. The simplest example of such scalars, which is called the minimal model, is an $SU(2)$ doublet of 2 complex scalar fields whose weak hypercharge is $Y_\phi = +1$,

$$\phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix}, \tag{4.29}$$

where φ^+ and φ^0 are positively charged and neutral complex scalar fields, respectively. The Lagrangian for these scalars is given by

$$\mathcal{L}_s = (D_\mu\phi)^\dagger(D^\mu\phi) - V(\phi^\dagger\phi), \tag{4.30}$$

with

$$D_\mu\phi = (\partial_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{A}_\mu - \frac{i}{2}g'B_\mu)\phi, \tag{4.31}$$

where the explicit form of the covariant derivative is due to $Y_\phi = +1$. The potential term $V(\phi^\dagger\phi)$ being gauge invariant is given by

$$V(\phi^\dagger\phi) = m^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2, \tag{4.32}$$

where m^2 and λ are real constant parameters. λ should be positive to ensure the stable vacuum. Further higher power terms of $\phi^\dagger\phi$ are not allowed in order for the theory to be renormarizable.

We can also add the coupling terms between fermions and scalars, called Yukawa interaction terms, which are $SU(2)_L \times U(1)_Y$ gauge invariant and are to provide the electron mass after the spontaneous symmetry breakdown,

$$\mathcal{L}_Y = -G_e(\bar{L}\phi R + \bar{R}\phi^\dagger L) + h.c., \tag{4.33}$$

where G_e is called Yukawa coupling constant and cannot be determined within the GWS model itself. One can easily check the $SU(2)_L \times U(1)_Y$ invariance of $\mathcal{L}_Y$ using the values of hypercharge of L, R and ϕ defined above.

A full set of $SU(2)_L \times U(1)_Y$ gauge invariant Lagrangian of the GWS model is, thus, given by the sum of pieces presented above,

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_G + \mathcal{L}_s + \mathcal{L}_Y. \tag{4.34}$$

Strictly speaking, the gauge fixing term with gauge parameter ξ (of "R_ξ gauge") and the associated term for Faddeev-Popov ghosts also should be included, in the quantization procedure of the theory. (See, for example, the books by Peskin and Schroeder (Peskin and Schroeder, 1995) or Pokorski (Pokorski, 2000).)

4.3 Spontaneous Breaking of $SU(2)_L \times U(1)_Y$ Symmetry

As shown in Fig. 3.2(b), the potential $V(\phi^\dagger\phi)$ of (4.32) with positive λ and negative $m^2 = -\mu^2(\mu^2 > 0)$ has a minimum at the value of ϕ determined by

$$\phi^\dagger\phi = |\phi|^2 = \frac{v^2}{2}, \quad \text{with} \quad v = \sqrt{\frac{\mu^2}{\lambda}}. \tag{4.35}$$

Then, as described in Chapter 3, spontaneous symmetry breaking occurs when the scalar doublet ϕ of (4.29) develops a vacuum expectation value

$$\phi_0 = \langle 0|\phi|0\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}. \tag{4.36}$$

It should be noted that though T^3 and Y do not annihilate the vacuum ϕ_0

$$T^3\phi_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = -\frac{1}{2}\phi_0, \tag{4.37}$$

$$Y\phi_0 = \phi_0, \tag{4.38}$$

and thus they are broken generators, i.e. $e^{-i\alpha^3 T^3}\phi_0 \neq \phi_0$ and $e^{-i\beta\frac{Y}{2}}\phi_0 \neq \phi_0$, the electric charge operator Q is not a broken generator,

$$Q\phi_0 = (T^3 + \frac{Y}{2})\phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = 0, \tag{4.39}$$

i.e. $e^{-i\varepsilon Q}\phi_0 = \phi_0$, where ε is an arbitrary parameter. Therefore, even after the symmetry breaking, there remains a symmetry associated with the charge operator Q of $U(1)_{em}$ being compatible with our real world.

Now, it is convenient to parametrize the scalar doublet with 4 degrees of freedom in terms of the fields denoting the shifts from the vacuum state ϕ_0,

$$\phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} = e^{i\vec{\tau}\cdot\vec{\xi}/2v}\begin{pmatrix} 0 \\ (v+H)/\sqrt{2} \end{pmatrix}. \tag{4.40}$$

Here the original 2 complex scalar fields φ^+ and φ^0 in (4.29) are replaced by 4 real fields, $\xi_i (i = 1, 2, 3)$ and H, where ξ_i are so-called Goldstone bosons being absorbed into the longitudinal components of $W^\pm$ and Z^0 bosons as described later and H is a Higgs boson. (4.36) leads to zero vacuum expectation values for all of these fields

$$\langle 0|\xi_i|0\rangle = \langle 0|H|0\rangle = 0. \quad (i = 1, 2, 3) \tag{4.41}$$

Here we can rewrite the Lagrangian in the "unitary gauge", where 3 Goldstone bosons ξ_i disappear by being 'eaten up' by gauge bosons, $W^{\pm}$ and Z^0, and thus, physical particle spectra and their interactions become apparent. By applying the unitary $SU(2)$ transformation

$$U(\xi) = e^{-i\vec{\tau}\cdot\vec{\xi}/2v}, \tag{4.42}$$

one can come to the real world induced in the unitary gauge. Then, we can define the new fields in our real world as

$$\begin{aligned}
\phi' &= U(\xi)\phi = \begin{pmatrix} 0 \\ (v+H)/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}(v+H)\chi, &(4.43)\\
L' &= U(\xi)L, &(4.44)\\
\vec{\mathbf{A}}'_\mu &= U(\xi)\vec{\mathbf{A}}_\mu U(\xi)^{-1} - \frac{i}{g}(\partial_\mu U(\xi))U^\dagger(\xi), &(4.45)
\end{aligned}$$

with $\chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{A}}_\mu = \vec{A}_\mu \cdot \frac{\vec{\tau}}{2}$, where the new fields transformed from the original ones are presented with a prime. R and B_μ remain unchanged under this $SU(2)$ transformation,

$$\begin{aligned}
R' &= R, &(4.46)\\
B'_\mu &= B_\mu. &(4.47)
\end{aligned}$$

The Lagrangian is invariant under this transformation and one can rewrite each piece as

$$\begin{aligned}
\mathcal{L}_F &= \bar{L}' i\gamma^\mu(\partial_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{A}'_\mu + \frac{i}{2}g'B'_\mu)L' + \bar{R}' i\gamma^\mu(\partial_\mu + ig'B'_\mu)R', &(4.48)\\
\mathcal{L}_G &= -\frac{1}{4}F'^{i}_{\mu\nu}F'^{i\mu\nu} - \frac{1}{4}B'_{\mu\nu}B'^{\mu\nu}, &(4.49)\\
\mathcal{L}_s &= (D_\mu\phi)'(D^\mu\phi)' - V(\phi'^\dagger\phi'), &(4.50)\\
\mathcal{L}_Y &= -G_e(\bar{L}'\phi' R' + \bar{R}'\phi'^\dagger L') + h.c. &(4.51)
\end{aligned}$$

Now let us discuss the physics described by this Lagrangian realized in the unitary gauge. First we consider the scalar sector. The scalar fields generate masses of gauge bosons and those of quarks and leptons via the Higgs mechanism. $\mathcal{L}_s$ is explicitly written as

$$\mathcal{L}_s = (D_\mu\phi)'(D^\mu\phi)' - V(\phi'^\dagger\phi'), \tag{4.52}$$

with

$$\begin{aligned}
(D_\mu\phi)' &= (\partial_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{A}'_\mu - \frac{i}{2}g'B'_\mu)\phi'\\
&= (\partial_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{A}'_\mu - \frac{i}{2}g'B'_\mu)\frac{1}{\sqrt{2}}(v+H)\chi. &(4.53)
\end{aligned}$$

The first term of (4.52) contains the mass-squared term for weak gauge bosons which is originated from the quadratic terms of gauge fields as shown in the following,

$$\begin{aligned}
\mathcal{L}_{mass} &= \frac{v^2}{2}\chi^\dagger(g\frac{\vec{\tau}}{2}\cdot\vec{A}'_\mu + \frac{g'}{2}B'_\mu)(g\frac{\vec{\tau}}{2}\cdot\vec{A}'^\mu + \frac{g'}{2}B'^\mu)\chi \\
&= \frac{v^2}{8}(g^2\vec{A}'_\mu\cdot\vec{A}'^\mu + g'^2 B'_\mu B'^\mu - 2gg'B'_\mu A'^{3\mu}) \\
&= \frac{v^2}{8}(g^2 A'^1_\mu A'^{1\mu} + g^2 A'^2_\mu A'^{2\mu} + (gA'^3_\mu - g'B'_\mu)^2), \qquad (4.54)
\end{aligned}$$

where in turning from the 1st line to the 2nd line, we used the formula $\tau^i\tau^j = \delta_{ij} + i\varepsilon^{ijk}\tau^k$. Now let us introduce charged boson fields $W^\pm$ defined by

$$W^\pm_\mu = \frac{A'^1_\mu \mp iA'^2_\mu}{\sqrt{2}}. \qquad (4.55)$$

Then the sum of the 1st and 2nd terms of (4.54) can be written as $\frac{1}{4}g^2v^2W^+_\mu W^{-\mu}$. It means that the charged vector bosons $W^\pm$ are massive with the mass

$$M_W = \frac{1}{2}gv. \qquad (4.56)$$

The remaining term which is described by neutral fields can be written as

$$\frac{v^2}{8}(A'^3_\mu \ B'_\mu)\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} A'^{3\mu} \\ B'^\mu \end{pmatrix}, \qquad (4.57)$$

which can be diagonalized into

$$\frac{v^2}{8}(Z_\mu \ A_\mu)\begin{pmatrix} g^2+g'^2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Z^\mu \\ A^\mu \end{pmatrix} = \frac{v^2}{8}(g^2+g'^2)Z_\mu Z^\mu + 0\cdot A_\mu A^\mu, \qquad (4.58)$$

by an orthogonal transformation

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}\begin{pmatrix} A'^3_\mu \\ B'_\mu \end{pmatrix}, \qquad (4.59)$$

where θ_W is called the weak mixing angle or Weinberg angle. The diagonalization leads to

$$\tan\theta_W = \frac{g'}{g} \qquad (4.60)$$

or

$$\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}, \quad \cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}. \qquad (4.61)$$

From (4.58), we see that the neutral Z boson becomes massive with the mass

$$M_Z = \frac{1}{2}v\sqrt{g^2+g'^2}, \qquad (4.62)$$

and another neutral boson A_μ is massless and hence can be identified with the real photon. Note that in the GWS model the mass of Z^0 boson is related to the one of $W^\pm$ bosons as

$$M_Z = \frac{M_W}{\cos\theta_W}. \tag{4.63}$$

One can see that the masses of $W^\pm$ and Z^0 are quite large; in fact by using (4.83), (4.84) and (4.89) below, we can estimate them to be

$$\begin{aligned} M_W &= \frac{1}{2}gv = \frac{1}{2}\left(\frac{e^2}{\sqrt{2}G_F}\right)^{1/2}\frac{1}{\sin\theta_W} \simeq \frac{38}{\sin\theta_W}\text{GeV} > 37\text{GeV}, & (4.64)\\ M_Z &= \frac{M_W}{\cos\theta_W} \simeq \frac{76}{\sin 2\theta_W}\text{GeV} > 76\text{GeV}. & (4.65)\end{aligned}$$

The values of M_W and M_Z are obtained if $\sin\theta_W$ is determined experimentally. Actually, the value of $\sin^2\theta_W$ is obtained in experiment to be around 0.23, leading to $M_W \simeq 80$GeV and $M_Z \simeq 90$GeV.

The potential term (4.32) of scalars becomes, after symmetry breaking,

$$\begin{aligned} V(\phi'^\dagger\phi') &= -\frac{\mu^2}{2}(v+H)^2\chi^\dagger\chi + \frac{\lambda}{4}(v+H)^4(\chi^\dagger\chi)^2 \\ &= -\frac{\mu^2v^2}{4} + \frac{1}{2}(2\mu^2)H^2 + \lambda vH^3 + \frac{\lambda}{4}H^4. \end{aligned} \tag{4.66}$$

From (4.66), we see that the mass of the physical Higgs boson H can be identified with

$$M_H = \sqrt{2\mu^2}, \tag{4.67}$$

whose value cannot be predicted, based on some principle, in the GWS model. After all, in the unitary gauge the Lagrangian $\mathcal{L}_s$ results in (up to a constant term)

$$\begin{aligned} \mathcal{L}_s &= (D_\mu\phi)'(D^\mu\phi)' - V(\phi'^\dagger\phi') \\ &= \frac{1}{2}\partial_\mu H\partial^\mu H - \frac{1}{2}M_H^2H^2 - \lambda vH^3 - \frac{\lambda}{4}H^4 \\ &+ \frac{g^2}{8}(H^2+2Hv)\left[\frac{1}{\cos^2\theta_W}Z_\mu Z^\mu + 2W_\mu^+W^{-\mu}\right] \\ &+ M_W^2W_\mu^+W^{-\mu} + \frac{1}{2}M_Z^2Z_\mu Z^\mu. \end{aligned} \tag{4.68}$$

Let us consider next the Yukawa interaction term $\mathcal{L}_Y$ in the unitary gauge

$$\begin{aligned} \mathcal{L}_Y &= -G_e(\bar{L}'\phi'R' + \bar{R}'\phi'^\dagger L') + h.c. \\ &= -G_e\left(\bar{e}'_L\frac{1}{\sqrt{2}}(v+H)e'_R + \bar{e}'_R\frac{1}{\sqrt{2}}(v+H)e'_L\right) + h.c. \\ &= -\frac{G_ev}{\sqrt{2}}\bar{e}'e' - \frac{G_e}{\sqrt{2}}H\bar{e}'e'. \end{aligned} \tag{4.69}$$

Since only physical degrees of freedom appear in the unitary gauge, we can identify e' to be physical electron and thus we suppress the prime $'$ hereafter. Then, the first term of this equation corresponds to the mass term of an electron with the mass

$$m_e = \frac{G_e}{\sqrt{2}} v. \tag{4.70}$$

It is interesting to note that the electron mass is proportional to the vacuum expectation value v of the Higgs boson as well as in the case of weak gauge boson masses. The second term shows an interaction term of an electron to the Higgs boson with the coupling constant

$$\frac{G_e}{\sqrt{2}} = \frac{m_e}{v}. \tag{4.71}$$

It is remarkable that the coupling constant is proportional to the electron mass.

In fact, the Higgs coupling to fermions in the Feynman rule (see Appendix B.3) is given by $-i\frac{m_f}{v} = -i\frac{e}{2\sin\theta_W}\frac{m_f}{M_W} = -i(\sqrt{2}G_F)^{1/2}m_f$ from (4.56), (4.89) and (4.83), where G_F is the Fermi coupling constant and m_f is the mass of the fermion coupled to a Higgs boson. Since $v \simeq 246$GeV (see (4.85)), the Higgs coupling to ordinary quarks (u, d, s, c, b) or leptons (ν_e, e, ν_μ, μ, ν_τ, τ) is extremely small and one can neglect the Higgs interaction effects on these particles. However, if there exist extremely heavy fermions such as the top quark, 4-th generation quarks (Q) or leptons (L), the Higgs coupling to those particles becomes large and cannot be neglected. (Inazawa and Morii, 1988; Strassler and Peskin, 1991) It might be even larger than gluon coupling to those extremely heavy quarks.

4.4 Charged and Neutral Currents, Comparison with Effective Fermi Theory

The fermion part $\mathcal{L}_F$ of (4.48) in the unitary gauge is written as

$$\mathcal{L}_F = \bar{L}' i\gamma^\mu\partial_\mu L' + \bar{R}' i\gamma^\mu\partial_\mu R' + g\vec{J}_\mu \cdot \vec{A}'^\mu + \frac{g'}{2} J_\mu^Y B'^\mu, \tag{4.72}$$

where

$$\begin{aligned} \vec{J}_\mu &= \bar{L}'\gamma_\mu \frac{\vec{\tau}}{2} L', &(4.73)\\ J_\mu^Y &= -\bar{L}'\gamma_\mu L' - 2\bar{R}'\gamma_\mu R'. &(4.74) \end{aligned}$$

In (4.72), the 1st and 2nd terms result in the conventional kinetic terms of an electron and its neutrino

$$\bar{L}' i\gamma_\mu\partial_\mu L' + \bar{R}' i\gamma^\mu\partial_\mu R' = \bar{e} i\gamma^\mu\partial_\mu e + \bar{\nu}_{eL} i\gamma_\mu\partial_\mu \nu_{eL}, \tag{4.75}$$

where the prime for electron and neutrino fields is dropped in the right hand side.

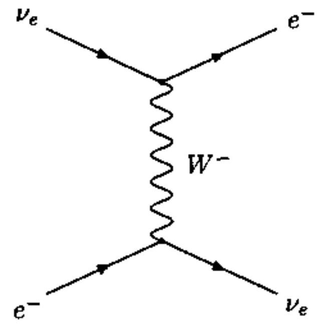


Fig. 4.1

From (4.72), one can pick up the charged current interaction which is given by

$$\mathcal{L}_{CC} = g(J^1_\mu A'^{1\mu} + J^2_\mu A'^{2\mu}) = \frac{g}{\sqrt{2}}(J^-_\mu W^{-\mu} + J^+_\mu W^{+\mu}), \tag{4.76}$$

where

$$J^\pm_\mu = J^1_\mu \pm iJ^2_\mu. \tag{4.77}$$

$J^\pm_\mu$ are explicitly expressed as

$$J^+_\mu = J^1_\mu + iJ^2_\mu = \bar{L}'\gamma_\mu\tau^+L' = \bar{\nu}_{eL}\gamma_\mu e_L = \frac{1}{2}\bar{\nu}_e\gamma_\mu(1-\gamma_5)e, \tag{4.78}$$

$$J^-_\mu = J^1_\mu - iJ^2_\mu = \bar{L}'\gamma_\mu\tau^-L' = \bar{e}_L\gamma_\mu\nu_{eL} = \frac{1}{2}\bar{e}\gamma_\mu(1-\gamma_5)\nu_e. \tag{4.79}$$

Now, it is interesting to compare these charged currents with current $\times$ current interactions in the effective Fermi theory discussed in Chapter 2. Let us consider the lowest order process of $\nu_e + e \to \nu_e + e$ in the low energy, which occurs through an exchange of $W^\pm$ boson (Fig. 4.1). The corresponding Feynman amplitude is given by

$$M = -\frac{g^2}{2}J^{+\mu}\frac{i(-g_{\mu\nu} + \frac{q_\mu q_\nu}{M_W^2})}{q^2 - M_W^2 + i\epsilon}J^{-\nu}, \tag{4.80}$$

which reduces to

$$M = -i\frac{g^2}{2M_W^2}J^{+\mu}J^-_\mu, \tag{4.81}$$

in the limit of $q^2/M_W^2 \to 0$. The same amplitude can be obtained from the effective Fermi theory as

$$\mathcal{L}^{eff}_{CC} = -\frac{G_F}{\sqrt{2}}\cdot 4J^{+\mu}J^-_\mu = -\frac{G_F}{\sqrt{2}}\left(\bar{\nu}_e\gamma^\mu(1-\gamma_5)e\right)\left(\bar{e}\gamma_\mu(1-\gamma_5)\nu_e\right). \tag{4.82}$$

Therefore, one can see that the Fermi theory of weak interactions is the low energy effective theory of the GWS theory. Comparing (4.81) with (4.82), we obtain

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}. \tag{4.83}$$

By using

$$G_F = 1.16639 \times 10^{-5} GeV^{-2} \tag{4.84}$$

and (4.56), (4.83) and (4.84), one can determine the vacuum expectation value of ϕ as

$$v = (\sqrt{2}G_F)^{-\frac{1}{2}} \simeq 246 GeV, \tag{4.85}$$

which is called weak scale.

From (4.72), by using (4.59) one can also extract the neutral current piece

$$\begin{aligned} \mathcal{L}_{NC} &= gJ_\mu^3 A'^{3\mu} + \frac{1}{2}g' J_\mu^Y B'^\mu \\ &= (g\sin\theta_W J_\mu^3 + g'\cos\theta_W \frac{J_\mu^Y}{2})A^\mu \\ &+ (g\cos\theta_W J_\mu^3 - g'\sin\theta_W \frac{J_\mu^Y}{2})Z^\mu. \end{aligned} \tag{4.86}$$

Recalling (4.20), we have

$$J_\mu^{em} = J_\mu^3 + \frac{J_\mu^Y}{2}. \tag{4.87}$$

By using this relation, the current coupled to A^μ in (4.86) becomes

$$g\sin\theta_W J_\mu^3 + g'\cos\theta_W \frac{J_\mu^Y}{2} = g'\cos\theta_W J_\mu^{em} + (g\sin\theta_W - g'\cos\theta_W)J_\mu^3. \tag{4.88}$$

The second term vanishes from (4.60). The first term can be identified with the interaction of an electron to photon and hence we can identify the coupling constant to be electric charge e, i.e.

$$e = g'\cos\theta_W = g\sin\theta_W, \tag{4.89}$$

or

$$\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2}. \tag{4.90}$$

The current coupled to Z^μ becomes

$$g\cos\theta_W J_\mu^3 - g'\sin\theta_W \frac{J_\mu^Y}{2} = \frac{g}{\cos\theta_W}(J_\mu^3 - \sin^2\theta_W J_\mu^{em}), \tag{4.91}$$

and thus we define a new neutral current coupled to Z^μ as

$$J^Z_\mu = J^3_\mu - \sin^2\theta_W J^{em}_\mu. \tag{4.92}$$

In general, if a fermion family is given by

$$L = \begin{pmatrix} f \\ f' \end{pmatrix}_L, \; R^f = f_R, \; R^{f'} = f'_R, \tag{4.93}$$

J^Z_μ is explicitly written, by using (4.11) and (4.17), as

$$\begin{aligned} J^Z_\mu &= J^3_\mu - \sin^2\theta_W J^{em}_\mu \\ &= \bar{L}\gamma_\mu \frac{\tau^3}{2} L - \sin^2\theta_W (Q_f \bar{f}\gamma_\mu f + Q'_f \bar{f}'\gamma_\mu f') \\ &= a^f_L \bar{f}_L \gamma_\mu f_L + a^f_R \bar{f}_R\gamma_\mu f_R + a^{f'}_L \bar{f}'_L\gamma_\mu f'_L + a^{f'}_R \bar{f}'_R \gamma_\mu f'_R, \end{aligned} \tag{4.94}$$

where Q_f, Q'_f are electric charges of f, f' (in the unit of e), respectively, and

$$a^f_L = \frac{1}{2} - Q_f \sin^2\theta_W, \qquad a^{f'}_L = -\frac{1}{2} - Q'_f \sin^2\theta_W, \tag{4.95}$$

$$a^f_R = -Q_f \sin^2\theta_W, \qquad a^{f'}_R = -Q'_f \sin^2\theta_W. \tag{4.96}$$

Or, J^Z_μ is often written in literature as

$$J^Z_\mu = \bar{f}\gamma_\mu (C^f_V - C^f_A \gamma_5) f + \bar{f}'\gamma_\mu (C^{f'}_V - C^{f'}_A \gamma_5) f', \tag{4.97}$$

with

$$C^f_V = \frac{1}{2}(a^f_L + a^f_R) = \frac{1}{4} - Q_f \sin^2\theta_W, \tag{4.98}$$

$$C^f_A = \frac{1}{2}(a^f_L - a^f_R) = \frac{1}{4}, \tag{4.99}$$

$$C^{f'}_V = \frac{1}{2}(a^{f'}_L + a^{f'}_R) = -\frac{1}{4} - Q'_f \sin^2\theta_W, \tag{4.100}$$

$$C^{f'}_A = \frac{1}{2}(a^{f'}_L - a^{f'}_R) = -\frac{1}{4}. \tag{4.101}$$

Similarly as in (4.81), the lowest order amplitude for the low energy neutral current processes, e.g. $\nu_e + e \to \nu_e + e$, shown in Fig. 4.2, is given by

$$M = -i\frac{1}{2!}\frac{g^2}{M_Z^2 \cos^2\theta_W} J^Z_\mu J^{Z\mu}. \tag{4.102}$$

where the factor $\frac{1}{2!}$ comes from the fact that $J^Z_\mu J^{Z\mu}$ is the square of the identical current. Thus, the corresponding effective Lagrangian for neutral current processes can be written as

$$\mathcal{L}^{eff}_{NC} = -\frac{G_N}{\sqrt{2}} \cdot 4 J^Z_\mu J^{Z\mu}, \tag{4.103}$$

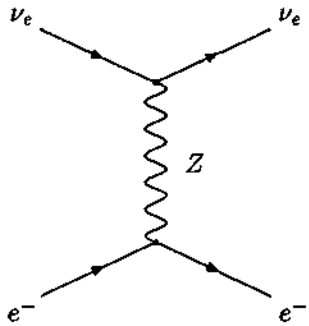


Fig. 4.2

defining $\frac{G_N}{\sqrt{2}} = \frac{g^2}{8M_Z^2 \cos^2\theta_W}$. Hence, we see that the total effective Lagrangian at low energy can be described by

$$\mathcal{L}^{eff} = -\frac{G_F}{\sqrt{2}} \cdot 4(J_\mu^+ J^{-\mu} + \rho J_\mu^Z J^{Z\mu}), \tag{4.104}$$

where

$$\rho = \frac{G_N}{G_F} = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1. \tag{4.105}$$

It should be noted that $\rho = 1$ is the inevitable consequence of the fact that the Higgs scalars belong to the doublet representation of $SU(2)_L$. It is no longer the case for Higgs scalars with higher dimensional representations of $SU(2)_L$, such as triplet. To see this explicitly, let us consider the case with higher representation of Higgs multiplet. Then, we can only replace the Higgs doublet ϕ in $\mathcal{L}_s$ of (4.52) by a new Higgs scalars $\phi_h'(x)$ of higher multiplet. The gauge boson mass term can be written as well as in (4.54)

$$\mathcal{L}_{mass} = \frac{v_h^2}{2}\chi_h^\dagger(g\vec{T}_h \cdot \vec{A}_\mu' + g'\frac{Y_h}{2}B_\mu)(g\vec{T}_h \cdot \vec{A}_\mu' + g'\frac{Y_h}{2}B_\mu)\chi_h, \tag{4.106}$$

where $\vec{T}_h$ and Y_h are appropriate generators corresponding to $SU(2)_L$ and $U(1)_Y$, respectively, and v_h is the vacuum expectation value of this Higgs scalar. χ_h represents a similar vector as in (4.43), in which only one nonzero component is 1 and others are zero. For this multiplet we have again $Q_h = T_{h3} + \frac{Y_h}{2}$, as before, where Q_h is charge of h and T_{h3} is the 3rd component of $\vec{T}_h$. Now, let us put $Q_h = 0$ for one component of $\phi_h'(x)$. For this component, we have $\frac{Y_h}{2} = -T_{h3}$. Then, the vacuum expectation value v_h of this field generates the masses of gauge bosons, except the photon, in the same way as the case of the doublet Higgs scalars described before. The resultant mass term is

$$\mathcal{L}_{mass} = \frac{v_h^2}{2}\chi_h^\dagger\left(g^2\vec{T}_h \cdot \vec{A}_\mu'\vec{T}_h \cdot \vec{A}'^\mu + g'^2(\frac{Y_h}{2})^2 B_\mu' B'^\mu + 2gg'\vec{T}_h \cdot \vec{A}_\mu'\frac{Y_h}{2}B'^\mu\right)\chi_h$$

$$\begin{aligned}
&= \frac{g^2 v_h^2}{2}(A'^1_\mu A'^{1\mu} T^2_{h1} + A'^2_\mu A'^{2\mu} T^2_{h2}) \\
&+ \frac{g^2 v_h^2}{2}\left(A'^3_\mu A'^{3\mu} T^2_{h3} + \frac{g'^2}{g^2} T^2_{h3} B'_\mu B'^\mu - 2\frac{g'}{g} T^2_{h3} A'^3_\mu B'^\mu\right) \\
&= \frac{g^2 v_h^2}{2}(A'^1_\mu A'^{1\mu} T^2_{h1} + A'^2_\mu A'^{2\mu} T^2_{h2}) + \frac{g^2 v_h^2}{2\cos^2\theta_W} Z^2_\mu T^2_{h3},
\end{aligned} \tag{4.107}$$

where in the second and the third lines, $\chi_h^\dagger T^2_{hi}\chi_h$ $(i = 1, 2, 3)$ has been written simply as T^2_{hi}. From this equation, we have

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = \frac{\sum_h v_h^2 T^2_{h\perp}}{\sum_h v_h^2 T^2_{h3}}, \tag{4.108}$$

with

$$T^2_{h\perp} = \frac{1}{2}(T^2_{h1} + T^2_{h2}) = \frac{1}{2}(\vec{T}_h^2 - T^2_{h3}) = \frac{1}{2}\left(I_h(I_h+1) - T^2_{h3}\right), \tag{4.109}$$

where I_h is the (magnitude of) weak isospin of h. Therefore, ρ can take any value, in principle. When all Higgs multiplets belong to the same representation, as in the case of the supersymmetric extension of the standard model, i.e. the "Minimal Supersymmetric Standard Model" (MSSM), $\sum_i v_h^2$ cancel out between the numerator and the denominator of (4.108) and we have

$$\rho = \frac{1}{2}\frac{I_h(I_h+1) - T^2_{h3}}{T^2_{h3}}. \tag{4.110}$$

One can easily check $\rho = 1$ for $I_h = \frac{1}{2}$ and $T_{h3} = \pm\frac{1}{2}$, as in the case of the standard model or MSSM. In other words, the GWS model with minimal Higgs doublet necessarily predicts $\rho = 1$ at the classical (tree) level, though the ρ deviates from 1 at the quantum level, as we will see in Chapter 8.

4.5 Addition of More Leptons

So far we have considered only one lepton family, i.e. electron e and its neutrino ν_e. The GWS model can be easily extended for the case with more lepton families. Let us first consider, for simplicity, the case of 2 lepton families, electron (1st generation) and muon (2nd generation) families, represented as

$$L_e = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad R_e = e_R; \quad L_\mu = \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad R_\mu = \mu_R. \tag{4.111}$$

(1st generation) (2nd generation)

The Lagrangian for this fermion sector is given by

$$\mathcal{L}_F = \mathcal{L}_F^{(e)} + \mathcal{L}_F^{(\mu)}, \tag{4.112}$$

where both terms have the same expression as (4.23) by replacing L and R with those given in (4.111), respectively. The Lagrangian for gauge field and scalar boson sector remains unchanged. However, as for the Yukawa interactions, new terms are allowed without contradicting with the requirement of $SU(2)_L \times U(1)_Y$ invariance,

$$\mathcal{L}_Y = -G_{ee}\bar{L}_e\phi R_e - G_{\mu\mu}\bar{L}_\mu\phi R_\mu - G_{e\mu}\bar{L}_e\phi R_\mu - G_{\mu e}\bar{L}_\mu\phi R_e + h.c. \tag{4.113}$$

Notice that here we require only the local gauge invariance and do not assume separately the electron number and muon number conservation to start with. After the spontaneous symmetry breaking, one can easily obtain the fermion mass term from $\mathcal{L}_Y$ just as we did before for an electron family.

$$\begin{aligned}\mathcal{L}_F^{(mass)} &= -(m_{ee}\bar{e}_Le_R + m_{\mu\mu}\bar{\mu}_L\mu_R + m_{e\mu}\bar{e}_L\mu_R + m_{\mu e}\bar{\mu}_Le_R + h.c.) \\ &= -(\bar{e}_L\ \bar{\mu}_L)\begin{pmatrix} m_{ee} & m_{e\mu} \\ m_{\mu e} & m_{\mu\mu}\end{pmatrix}\begin{pmatrix} e_R \\ \mu_R\end{pmatrix} + h.c. \end{aligned}\tag{4.114}$$

where $m_{ij} = G_{ij}\frac{v}{\sqrt{2}}$ $(i, j = e \text{ or } \mu)$.

Note that the mass matrix

$$M = \begin{pmatrix} m_{ee} & m_{e\mu} \\ m_{\mu e} & m_{\mu\mu}\end{pmatrix} \tag{4.115}$$

is, in general, complex and not necessarily hermitian and is not diagonalized by use of one unitary matrix. In general, it can be mathematically shown that an arbitrary $n \times n$ complex matrix M can be diagonalized by a bi-unitary transformation as

$$U^\dagger MV = M_d, \tag{4.116}$$

where M_d is a diagonal matrix and both of U and V are unitary matrices, which are the unitary matrices to diagonalize the hermitian matrices $MM^\dagger$ and $M^\dagger M$: $U^\dagger(MM^\dagger)U = V^\dagger(M^\dagger M)V = M_dM_d^*$. Redefining the phase of μ and e fields appropriately, one can always make the elements of M_d be real and positive as

$$U^\dagger MV = M_d = \begin{pmatrix} m_e & 0 \\ 0 & m_\mu\end{pmatrix}, \tag{4.117}$$

where m_e and m_μ correspond to the masses of electron and muon, respectively.

To follow this program, let us diagonalize the M of (4.115) by applying the unitary matrices,

$$U = \begin{pmatrix} \cos\theta_L & -\sin\theta_L \\ \sin\theta_L & \cos\theta_L\end{pmatrix}, \quad V = \begin{pmatrix} \cos\theta_R & -\sin\theta_R \\ \sin\theta_R & \cos\theta_R\end{pmatrix}. \tag{4.118}$$

To be precise, each of U and V may have a few phases, in addition to the mixing angle, θ_L or θ_R. These phases, however, turn out to be physically irrelevant (see the discussion in the subsection 4.6.2 below). Thus here we simply ignore those phases.

Then the $\mathcal{L}_F^{(mass)}$ can be rewritten as

$$\begin{aligned}\mathcal{L}_F^{(mass)} &= -(\bar{e}_L\ \bar{\mu}_L)M\begin{pmatrix} e_R \\ \mu_R \end{pmatrix} + h.c.\\ &= -(\bar{e}'_L\ \bar{\mu}'_L)M_d\begin{pmatrix} e'_R \\ \mu'_R \end{pmatrix} + h.c. \end{aligned}\tag{4.119}$$

where

$$\begin{pmatrix} e'_L \\ \mu'_L \end{pmatrix} = U^\dagger\begin{pmatrix} e_L \\ \mu_L \end{pmatrix} = \begin{pmatrix} \cos\theta_L & \sin\theta_L \\ -\sin\theta_L & \cos\theta_L \end{pmatrix}\begin{pmatrix} e_L \\ \mu_L \end{pmatrix}, \tag{4.120}$$

$$\begin{pmatrix} e'_R \\ \mu'_R \end{pmatrix} = V^\dagger\begin{pmatrix} e_R \\ \mu_R \end{pmatrix} = \begin{pmatrix} \cos\theta_R & \sin\theta_R \\ -\sin\theta_R & \cos\theta_R \end{pmatrix}\begin{pmatrix} e_R \\ \mu_R \end{pmatrix}. \tag{4.121}$$

Notice that the fields with prime $(e'_{L,R}, \mu'_{L,R})$ which are called mass eigenstates are not equal to the fields without prime $(e_{L,R}, \mu_{L,R})$ which are called current/weak eigenstates. The observed states are mass eigenstates. Therefore, when we write weak charged currents interaction in terms of the observed states, the interaction is no longer diagonal in the generation space and we potentially have an intergenerational mixing. The detail will be discussed later in section 4.6 when we discuss the quark sector.

However, here we just point out that in the case of massless neutrinos the mixing angle θ_L can be rotated away and thus, the mass eigenstates of leptons become equal to weak eigenstates. This can be seen considering the lepton doublet (4.111). By using (4.120), the weak eigenstates are written as

$$\begin{aligned} L_e &= \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L = \begin{pmatrix} \nu_{eL} \\ \cos\theta_L e'_L - \sin\theta_L \mu'_L \end{pmatrix},\\ L_\mu &= \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L = \begin{pmatrix} \nu_{\mu L} \\ \sin\theta_L e'_L + \cos\theta_L \mu'_L \end{pmatrix}. \end{aligned}\tag{4.122}$$

As is easily shown, under the following rotation between the lepton doublets

$$\begin{pmatrix} L_e \\ L_\mu \end{pmatrix} \to \begin{pmatrix} L'_e \\ L'_\mu \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} L_e \\ L_\mu \end{pmatrix}, \tag{4.123}$$

we have

$$\bar{L}_e i\gamma^\alpha D_\alpha L_e + \bar{L}_\mu i\gamma^\alpha D_\alpha L_\mu = \bar{L}'_e i\gamma^\alpha D_\alpha L'_e + \bar{L}'_\mu i\gamma^\alpha D_\alpha L'_\mu. \tag{4.124}$$

Then, if we choose $\alpha = \theta_L$, we can write

$$L'_e = \cos\theta_L L_e + \sin\theta_L L_\mu = \begin{pmatrix} \cos\theta_L \nu_{eL} + \sin\theta_L \nu_{\mu L} \\ e'_L \end{pmatrix}, \tag{4.125}$$

$$L'_\mu = -\sin\theta_L L_e + \cos\theta_L L_\mu = \begin{pmatrix} -\sin\theta_L \nu_{eL} + \cos\theta_L \nu_{\mu L} \\ \mu'_L \end{pmatrix}. \tag{4.126}$$

Here ν_e and ν_μ are degenerate in their masses, as they are both massless, and therefore we can redefine the observed neutrino fields as

$$\begin{aligned}\nu'_{eL} &= \cos\theta_L \nu_{eL} + \sin\theta_L \nu_{\mu L}, &(4.127)\\ \nu'_{\mu L} &= -\sin\theta_L \nu_{eL} + \cos\theta_L \nu_{\mu L}, &(4.128)\end{aligned}$$

and so we end up with the weak doublets, which are still diagonal in the base of mass eigenstates. The mass eigenstates are now written as

$$L'_e = \begin{pmatrix} \nu'_e \\ e' \end{pmatrix}_L, \quad L'_\mu = \begin{pmatrix} \nu'_\mu \\ \mu' \end{pmatrix}_L, \tag{4.129}$$

and there is no flavor mixing between electron and muon in the weak charged current interactions. Thus, if we define individual lepton numbers as n_e(electron number)$= +1$ for e'^-, ν'_e; $n_e = -1$ for e'^+, $\bar{\nu}'_e$ and n_μ(muon number)$= +1$ for μ'^-, ν'_μ; $n_\mu = -1$ for μ'^+, $\bar{\nu}'_\mu$, respectively, then it is naturally derived that both of the electron number n_e and the muon number n_μ are conserved separately. The GWS model automatically leads to the conclusion that the decay mode $\mu^\pm \to e^\pm + \gamma$ is strictly forbidden. It is remarkable to note that the flavor conservation was not put in by hand but originated from the massless nature of the neutrinos in the GWS model. In other words, if neutrinos are massive, one can no longer redefine the neutrino fields as in (4.127)–(4.128) and the weak eigenstates of neutrinos are given by a linear combination of the mass eigenstates ν_1 and ν_2, which leads to neutrino oscillations, as we will see in Chapter 6. (In the quark sector which will be discussed later in section 4.6, the mixing angle cannot be rotated away because quarks are massive with different masses.)

Let us next discuss the Higgs boson couplings to electrons and muons. From (4.113), one can extract them in the unitary gauge as

$$\mathcal{L}_{H\ell\ell} = -\frac{H}{v}(\bar{e}_L\ \bar{\mu}_L)\begin{pmatrix} m_{ee} & m_{e\mu} \\ m_{\mu e} & m_{\mu\mu} \end{pmatrix}\begin{pmatrix} e_R \\ \mu_R \end{pmatrix} + h.c. \tag{4.130}$$

When we go to the mass eigenstate by using (4.120) and (4.121), $\mathcal{L}_{H\ell\ell}$ turns into

$$\begin{aligned}\mathcal{L}_{H\ell\ell} &= -\frac{H}{v}(\bar{e}'_L\ \bar{\mu}'_L)M_d\begin{pmatrix} e'_R \\ \mu'_R \end{pmatrix} + h.c.\\ &= -\frac{H}{v}(m_e\bar{e}'e' + m_\mu\bar{\mu}'\mu'). &(4.131)\end{aligned}$$

This shows that (i)Higgs couplings do not mix the lepton flavors and (ii)they are proportional to the masses of leptons, which couple to the Higgs.

In conclusion, there is no lepton flavor number violation in the standard GWS model; the electron number n_e and muon number n_μ are strictly conserved. Extension of the model to tau lepton family (ν_τ, τ) is straightforward and the tau number

n_τ is also conserved. At present we have 3 generations of leptons;

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, e_R; \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \mu_R; \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L, \tau_R. \tag{4.132}$$

In the standard GWS model, n_e, n_μ, and n_τ are separately conserved. Therefore, the processes such as $\mu^- \to e^-\gamma$, $\mu^- \to e^+e^-e^-$ and $\mu^- + Z \to e^- + Z$, etc. are forbidden at the all orders of perturbation in the GWS model. Therefore, if some of these processes are discovered, it will clearly signal the presence of some physics beyond the standard model, such as MSSM.

4.6 Extension to Quarks

The well-established observation of left-handed charged weak currents of hadrons suggests that the left-handed components of quark fields should be constructed into a doublet similarly to the case of leptons. Now we have the following 3 generations of quarks.

$$\begin{array}{llll} \text{(1st generation)} & \begin{pmatrix} u \\ d \end{pmatrix}_L, & u_R, & d_R, \\ \text{(2nd generation)} & \begin{pmatrix} c \\ s \end{pmatrix}_L, & c_R, & s_R, \\ \text{(3rd generation)} & \begin{pmatrix} t \\ b \end{pmatrix}_L, & t_R, & b_R, \end{array} \tag{4.133}$$

The GWS model can be easily extended so that it can incorporate these quark families, though there are several differences between quarks and leptons; (i) quarks have three color degrees of freedom, whereas leptons are colorless. However, since the electroweak interactions are color-blind, one can simply suppress the color index of quarks in the GWS model, unless otherwise mentioned. (ii) all quarks are massive, while neutrinos are considered to be massless in the GWS model. Therefore, we have 2 right-handed singlets such as u_R and d_R for each generation, compared to the case of lepton families where we have only one right-handed singlet such as e_R. Thus, the Lagrangian must contain an additional term for the up-type right-handed singlets like u_R, in order to form their mass term. (iii) charge of quarks are different from that of leptons and thus, keeping the relation, $Q = T^3 + \frac{Y}{2}$, weak-hypercharges of quarks are also different from those of leptons. The charge Q, weak iso-spin (T, T^3) and weak-hypercharge Y of these quarks are summarized in Table 4.2.

To write the Lagrangian for quarks in a compact form, we introduce the following notation

$$Q_{Li} = \begin{pmatrix} U_i \\ D_i \end{pmatrix}_L, \quad U_{Ri}, \quad D_{Ri}, \quad (i = 1, 2, 3) \tag{4.134}$$

quark family	Q	(T, T^3)	Y
u_L, c_L, t_L	$+\frac{2}{3}$	$(\frac{1}{2}, +\frac{1}{2})$	$+\frac{1}{3}$
d_L, s_L, b_L	$-\frac{1}{3}$	$(\frac{1}{2}, -\frac{1}{2})$	$+\frac{1}{3}$
u_R, c_R, t_R	$+\frac{2}{3}$	0	$+\frac{4}{3}$
d_R, s_R, b_R	$-\frac{1}{3}$	0	$-\frac{2}{3}$

Table 4.2 Quantum numbers of quark families.

where i stands for the generation: $U_1 = u$, $U_2 = c$, $U_3 = t$, $D_1 = d$, $D_2 = s$, $D_3 = b$.

Based on Table 4.2, we can explicitly write down the Lagrangian for these quarks with covariant derivatives as

$$\begin{aligned}
\mathcal{L}_F &= \sum_{i=1}^{3} \bar{Q}_{Li} i\gamma^\mu (\partial_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{A}_\mu - \frac{i}{6}g'B_\mu)Q_{Li} \\
&+ \sum_{i=1}^{3} \bar{U}_{Ri} i\gamma^\mu (\partial_\mu - i\frac{2}{3}g'B_\mu)U_{Ri} \\
&+ \sum_{i=1}^{3} \bar{D}_{Ri} i\gamma^\mu (\partial_\mu + i\frac{1}{3}g'B_\mu)D_{Ri}. \qquad (4.135)
\end{aligned}$$

The terms for gauge fields $\mathcal{L}_G$ and scalar fields $\mathcal{L}_s$ remain unchanged from the case of leptons. As for the Yukawa coupling, we can write the following $SU(2)_L \times U(1)_Y$ invariant terms

$$\mathcal{L}_Y = -\sum_{i,j} (\Gamma_{i,j}^{(D)} \bar{Q}_{Li} \phi D_{Rj} + \Gamma_{i,j}^{(U)} \bar{Q}_{Li} \tilde{\phi} U_{Rj} + h.c.), \qquad (4.136)$$

where $\Gamma_{i,j}^{(D)}$ and $\Gamma_{i,j}^{(U)}$ are Yukawa couplings which are, in general, unconstrained complex parameters. $\tilde{\phi}$ whose weak-hypercharge is -1, i.e. $Y_{\tilde{\phi}} = -1$, is defined from ϕ as

$$\tilde{\phi} = i\tau_2 \phi^* = \begin{pmatrix} \varphi^{0*} \\ -\varphi^- \end{pmatrix}, \qquad (4.137)$$

where ϕ^* is the complex conjugate of ϕ given by (4.29). Inclusion of $\tilde{\phi}$ in the 2nd term of (4.136) is allowed because the fundamental representation and its conjugate are equivalent in $SU(2)$ group (see Appendix C3), though this is not the case for any other $SU(N)$ groups with $N > 3$. This may be easily seen as follows; Since $\phi^\dagger \phi$ is $SU(2)_L$ invariant, so is $(\phi^\dagger \phi)^t = \phi^t \phi^* = \phi^t(-i\tau_2)(i\tau_2\phi^*) = \phi^t \varepsilon \tilde{\phi}$ (ϵ: Levi-Cività tensor). On the other hand, anti-symmetric product of arbitrary two doublets is also $SU(2)_L$ singlet, which means that $\tilde{\phi}$ behaves as a doublet.

After spontaneous symmetry breaking, moving to the unitary gauge, we have

$$\phi \to \frac{1}{\sqrt{2}}(v+H)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.138}$$

$$\tilde{\phi} \to \frac{1}{\sqrt{2}}(v+H)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.139}$$

and we can see that the 1st and 2nd terms of (4.136) yield the mass matrices of the down-type and up-type quarks, respectively.

$$\begin{aligned} \bar{Q}_{Li}\phi D_{Rj} &\to (\bar{U}_{Li}\ \bar{D}_{Li})\frac{v+H}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} D_{Rj} \\ &= \frac{v}{\sqrt{2}}\bar{D}_{Li}D_{Rj} + \frac{H}{\sqrt{2}}\bar{D}_{Li}D_{Rj}, \end{aligned} \tag{4.140}$$

$$\begin{aligned} \bar{Q}_{Li}\tilde{\phi} U_{Rj} &\to (\bar{U}_{Li}\ \bar{D}_{Li})\frac{v+H}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} U_{Rj} \\ &= \frac{v}{\sqrt{2}}\bar{U}_{Li}U_{Rj} + \frac{H}{\sqrt{2}}\bar{U}_{Li}U_{Rj}. \end{aligned} \tag{4.141}$$

Here, all fields in the right-hand side are ones redefined in the unitary gauge.

4.6.1 *Quark mass matrix*

As described above, we can extract the quark mass matrices from $\mathcal{L}_Y$ of (4.136),

$$\mathcal{L}^{(m_Q)} = -\sum_{i,j}\bar{D}_{Li}\mathcal{M}^{(D)}_{i,j}D_{Rj} - \sum_{i,j}\bar{U}_{Li}\mathcal{M}^{(U)}_{i,j}U_{Rj} + h.c. \tag{4.142}$$

with

$$\mathcal{M}^{(D)}_{i,j} = \Gamma^{(D)}_{i,j}\frac{v}{\sqrt{2}}, \quad \mathcal{M}^{(U)}_{i,j} = \Gamma^{(U)}_{i,j}\frac{v}{\sqrt{2}}, \tag{4.143}$$

which are, in general, complex-valued matrices.

By applying the bi-unitary transformation to each term of (4.142) as we did in the lepton sector, we can diagonalize the mass matrices, $\mathcal{M}^{(D)}$ and $\mathcal{M}^{(U)}$, separately.

$$\mathcal{U}_D^\dagger \mathcal{M}^{(D)} \mathcal{V}_D = M^{(D)} \tag{4.144}$$

$$\mathcal{U}_U^\dagger \mathcal{M}^{(U)} \mathcal{V}_U = M^{(U)} \tag{4.145}$$

where $\mathcal{U}_{D,U}$ and $\mathcal{V}_{D,U}$ are unitary, and $M^{(D,U)}$ are diagonal matrices. Thus we have moved to the base of mass eigenstates:

$$D'_L = \mathcal{U}_D^\dagger D_L, \tag{4.146}$$

$$D'_R = \mathcal{V}_D^\dagger D_R, \tag{4.147}$$

$$U'_L = \mathcal{U}_U^\dagger U_L, \tag{4.148}$$
$$U'_R = \mathcal{V}_U^\dagger U_R, \tag{4.149}$$

where we used the matrix notation with

$$D_L = \begin{pmatrix} D_{L1} \\ D_{L2} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \quad D'_L = \begin{pmatrix} D'_{L1} \\ D'_{L2} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \tag{4.150}$$

etc. and the fields with and without prime denote the mass eigenstates and the weak eigenstates, respectively. Then, we have

$$\mathcal{L}^{(m_Q)} = -\bar{D}'_L M^{(D)} D'_R - \bar{D}'_R M^{(D)\dagger} D'_L - \bar{U}'_L M^{(U)} U'_R - \bar{U}'_R M^{(U)\dagger} U'_L. \tag{4.151}$$

Though $M^{(D)}$ and $M^{(U)}$ are diagonal matrices, they are complex, in general. However, if we parametrize the i-th diagonal element, using the absolute value $m_i^{(D,U)}$ and phase $\alpha_i^{(D,U)}$, as

$$M_i^{(D)} = m_i^{(D)} e^{i\alpha_i^{(D)}}, \tag{4.152}$$
$$M_i^{(U)} = m_i^{(U)} e^{i\alpha_i^{(U)}}, \tag{4.153}$$

we can absorb all these phase factors into new quark fields induced by a chiral $U(1)$ transformation, which keeps physics unchanged.

$$D'_{Li} \to e^{i\alpha_i^{(D)}/2} D'_{Li}, \tag{4.154}$$
$$D'_{Ri} \to e^{-i\alpha_i^{(D)}/2} D'_{Ri}, \tag{4.155}$$
$$U'_{Li} \to e^{i\alpha_i^{(U)}/2} U'_{Li}, \tag{4.156}$$
$$U'_{Ri} \to e^{-i\alpha_i^{(U)}/2} U'_{Ri}. \tag{4.157}$$

Then, finally the mass terms read as

$$\mathcal{L}^{(m_Q)} = -\bar{D}' m^{(D)} D' - \bar{U}' m^{(U)} U', \tag{4.158}$$

where $m^{(D,U)}$ are real and diagonal,

$$m^{(D)} = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \quad m^{(U)} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \tag{4.159}$$

and D' and U' are down-type and up-type quarks in the mass eigenstates, respectively.

$$D' = \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix}, \quad U' = \begin{pmatrix} u' \\ c' \\ t' \end{pmatrix}. \tag{4.160}$$

In (4.159), the matrix element m_i represents the quark mass of the i-th flavor.

In the above discussion on the GWS model, we have implicitly assumed that we have only electroweak interaction as the gauge interaction. However, if we take strong interactions of quarks into account, $\mathcal{L}^{(m_Q)}$ must have another piece induced by the QCD anomaly of the chiral $U(1)$ current; this is related to the problem called "strong CP problem". (See, for example, the review by Peccei (Peccei, 1989).)

4.6.2 *Flavor mixing*

4.6.2.1 *The case of charged current interaction*

Now let us discuss the charged weak current interactions, which can be extracted from the first line of (4.135)

$$\mathcal{L}_{CC}^{(Q)} = \frac{g}{\sqrt{2}} \sum_i \bar{U}_{Li} \gamma^\mu D_{Li} W_\mu^+ + \frac{g}{\sqrt{2}} \sum_i \bar{D}_{Li} \gamma^\mu U_{Li} W_\mu^- , \tag{4.161}$$

where the sum of i is taken for all generations. For a while, we let the number of generations, which we denote by n, be arbitrary, for generality. In terms of mass eigenstates defined in (4.146)–(4.149), we can write this $\mathcal{L}_{CC}^{(Q)}$ as

$$\mathcal{L}_{CC}^{(Q)} = \frac{g}{\sqrt{2}} \bar{U}_L' \mathcal{U}_L^\dagger \gamma^\mu \mathcal{U}_D D_L' W_\mu^+ + \frac{g}{\sqrt{2}} \bar{D}_L' \mathcal{U}_D^\dagger \gamma^\mu \mathcal{U}_L U_L' W_\mu^- , \tag{4.162}$$

where D_L' etc. are column vectors defined in (4.150), though now they are n-component vectors. Then, defining a flavor mixing matrix V, which is unitary,

$$V \equiv \mathcal{U}_U^\dagger \mathcal{U}_D, \tag{4.163}$$

we can write

$$\begin{aligned} \mathcal{L}_{CC}^{(Q)} &= \frac{g}{\sqrt{2}} \bar{U}_L' \gamma^\mu V D_L' W_\mu^+ + \frac{g}{\sqrt{2}} \bar{D}_L' V^\dagger \gamma^\mu U_L' W_\mu^- , \\ &= \frac{g}{2\sqrt{2}} \bar{U}' \gamma^\mu (1-\gamma_5) V D' W_\mu^+ + h.c. \end{aligned} \tag{4.164}$$

Here we are interested in how many physically meaningful independent parameters exist in V. V is a $n \times n$ unitary matrix with n^2 real parameters. (Because of the unitarity condition of V, there are n^2 relations among $2n^2$ parameters of V. Then the number of remaining parameters is n^2.) As is well-known, a real unitary matrix is an orthogonal matrix with $n(n-1)/2$ parameters. Then, V has $n(n-1)/2$ angles and $n^2 - n(n-1)/2 = n(n+1)/2$ phases. However, some of these phases can be absorbed into the new quark fields redefined by phase transformation; for example, when we consider

$$\begin{aligned} \bar{U}_L' \gamma^\mu V D_L' &= \bar{U}_{L1}' \gamma^\mu (V_{11} D_{L1}' + V_{12} D_{L2}' + \cdots + V_{1n} D_{Ln}') \\ &+ \bar{U}_{L2}' \gamma^\mu (V_{21} D_{L1}' + V_{22} D_{L2}' + \cdots + V_{2n} D_{Ln}') \end{aligned}$$

$$+ \quad \cdots\cdots\cdots$$
$$+ \quad \bar{U}'_{Ln}\gamma^{\mu}(V_{n1}D'_{L1} + V_{n2}D'_{L2} + \cdots + V_{nn}D'_{Ln}), \qquad (4.165)$$

the n components of the 1st column $(V_{11}, V_{21}, \cdots, V_{n1})^t$ of the mixing matrix V can be made real by writing $V_{i1} = |V_{i1}|e^{i\alpha_i^{(U)}}$ and redefining the up-type quark fields as $U'_{Li} \to e^{i\alpha_i^{(U)}}U'_{Li}$ $(i = 1, 2, \cdots, n)$. Similarly, we can make the $(n-1)$ components of the 1st row$(V_{12}, V_{13}, \cdots, V_{1n})$ real in the redefinition of $D'_{Li} \to e^{-i\alpha_i^{(D)}}D'_{Li}$ $(i = 2, 3, \cdots, n)$. (Note that $V_{11} = |V_{11}|$ has already been made real by the redefinition of U'_{L1}.) After this procedure, $(2n-1)$ phases are removed from V without changing physical consequences. After all, remaining parameters are characterized by $n(n-1)/2$ angles and $n(n+1)/2 - (2n-1) = (n-1)(n-2)/2$ phases.

- **Four quark case** $(n = 2)$

Here the quark mixing matrix V can be parametrized by only one flavor mixing angle θ_c, called Cabibbo angle. There is no phase $(\frac{(n-1)(n-2)}{2} = 0$ for $n = 2)$ and hence we have no CP violation in this case because of the absence of complex coupling constants. The charged weak current interaction can be written as

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}}(\bar{u}\ \bar{c})\gamma^{\mu}\frac{1-\gamma_5}{2}V\begin{pmatrix} d \\ s \end{pmatrix}W_{\mu}^{+} + h.c. \qquad (4.166)$$

with

$$V = \begin{pmatrix} V_{ud} & V_{us} \\ V_{cd} & V_{cs} \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}. \qquad (4.167)$$

Here and hereafter, the mass eigenstates of quarks are simply denoted by u, d, etc., suppressing the prime $'$.

This means that the left-handed quark doublets are written as

$$Q_L^{(1)} = \begin{pmatrix} u \\ d_c \end{pmatrix}_L, \quad Q_L^{(2)} = \begin{pmatrix} c \\ s_c \end{pmatrix}_L, \qquad (4.168)$$

where

$$\begin{pmatrix} d_c \\ s_c \end{pmatrix}_L = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}\begin{pmatrix} d \\ s \end{pmatrix}_L, \qquad (4.169)$$

and the weak charged current processes are due to the transition $u \leftrightarrow d_c$ or $c \leftrightarrow s_c$.

- **GIM mechanism**

In 1960s, only one quark doublet was known from the analysis of the β decay of neutron and Λ which occur via weak charged current transition $u \leftrightarrow d_c$ between the members of the doublet,

$$Q_L^{(1)} = \begin{pmatrix} u \\ d_c \end{pmatrix}_L = \begin{pmatrix} u \\ d\cos\theta_c + s\sin\theta_c \end{pmatrix}_L. \tag{4.170}$$

At that time, the orthogonal combination of d_L and s_L, i.e. $s_c = s\cos\theta_c - d\sin\theta_c$, was left as a $SU(2)_L$ singlet. Then, when we apply this to neutral currents, we have

$$\begin{aligned}\bar{Q}_L^{(1)}\gamma_\mu\frac{\tau^3}{2}Q_L^{(1)} &= \frac{1}{2}(\bar{u}_L\gamma_\mu u_L - \cos^2\theta_c\bar{d}_L\gamma_\mu d_L - \sin^2\theta_c\bar{s}_L\gamma_\mu s_L \\ &- \cos\theta_c\sin\theta_c(\bar{d}_L\gamma_\mu s_L + \bar{s}_L\gamma_\mu d_L)),\end{aligned} \tag{4.171}$$

where the final term corresponds to a strangeness changing neutral current. (We can check that the part of the neutral current, proportional to the electric charge, does not contain such strangeness changing neutral current.) If this term really exists, we necessarily predict a rather large decay width of the flavor changing neutral current transition such as $K_L^0 \to \mu^+\mu^-$ or $K^+ \to \pi^+\bar{\nu}\nu$, since flavor changing $\bar{s}dZ$ and $\bar{d}sZ$ vertices appear already at the classical (tree) level. But this prediction is completely in disagreement with experimental results, $Br(K_L^0 \to \mu^+\mu^-) = (7.25 \pm 0.16) \times 10^{-9}$, or $Br(K^+ \to \pi^+\bar{\nu}\nu) = (1.6^{+1.8}_{-0.8}) \times 10^{-10}$. (See Chapter 9 for some detailed discussions on these flavor (strangeness) changing neutral current processes.)

In 1970, Glashow, Iliopuolos and Maiani(GIM) (Glashow, Iliopuolos and Maiani, 1970) proposed a new mechanism to solve this problem by introducing the 2nd quark doublet which contains the 4-th quark, which is now called the charm quark c,

$$Q_L^{(2)} = \begin{pmatrix} c \\ s_c \end{pmatrix}_L = \begin{pmatrix} c \\ s\cos\theta_c - d\sin\theta_c \end{pmatrix}_L, \tag{4.172}$$

which produces additional neutral current,

$$\begin{aligned}\bar{Q}_L^{(2)}\gamma_\mu\frac{\tau^3}{2}Q_L^{(2)} &= \frac{1}{2}(\bar{c}_L\gamma_\mu c_L - \cos^2\theta_c\bar{s}\gamma_\mu s_L + \sin^2\theta_c\bar{d}_L\gamma_\mu d_L \\ &+ \cos\theta_c\sin\theta_c(\bar{d}_L\gamma_\mu s_L + \bar{s}_L\gamma_\mu d_L)).\end{aligned} \tag{4.173}$$

Summing up (4.171) and (4.173), we find that the neutral current finally becomes flavor diagonal;

$$\bar{Q}_L^{(1)}\gamma_\mu\frac{\tau^3}{2}Q_L^{(1)} + \bar{Q}_L^{(2)}\gamma_\mu\frac{\tau^3}{2}Q_L^{(2)} = \frac{1}{2}(\bar{u}_L\gamma_\mu u_L + \bar{c}_L\gamma_\mu c_L - \bar{d}_L\gamma_\mu d_L - \bar{s}_L\gamma_\mu s_L), \tag{4.174}$$

and we have no flavor changing neutral current at all at the tree level.

Then, how the flavor changing neutral current transitions are possible, even though it is extremely suppressed? GIM proposed a wonderful idea that it will occur at the quantum level via, for example, the higher order charged current interaction shown in Fig. 4.3 (for the $K_L^0 \to \mu^+\mu^-$ decay). The contribution of the diagram

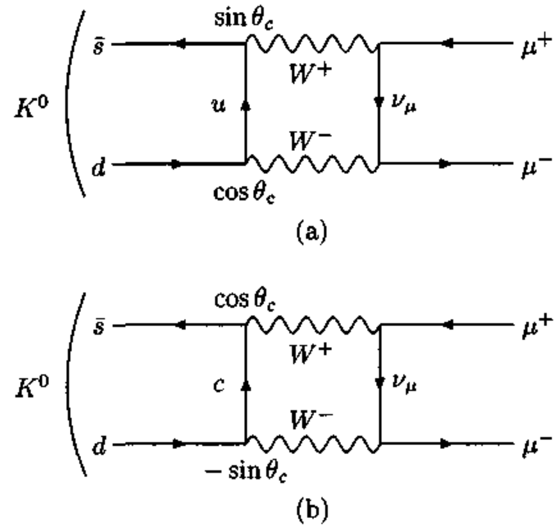


Fig. 4.3

(a) to the decay amplitude behave, ignoring the powers of small $\frac{m_u^2}{M_W^2}$ and $\frac{m_c^2}{M_W^2}$, as

$$A(K_L^0 \to \mu^+\mu^-) \propto \frac{g^4 \sin\theta_c \cos\theta_c}{M_W^2} \approx G_F \alpha \sin\theta_c \cos\theta_c, \tag{4.175}$$

where $\alpha = e^2/4\pi$ is the fine structure constant. But this contribution, though suppressed by the factor α, is still too large to account for the decay width of the "rare process". However, the contribution of the 2nd doublet introduced by GIM, as shown in the diagram (b), yields a contribution, $-\frac{g^4 \sin\theta_c \cos\theta_c}{M_W^2}$, which is just opposite to that of (a). Thus the amplitude exactly vanishes at the leading order of the perturbation in the powers of $\frac{m_q^2}{M_W^2}$ $(q = u, c)$, and therefore the non-vanishing contribution stems from the 1st order of $\frac{m_q^2}{M_W^2}$, yielding a suppression factor $\frac{m_c^2 - m_u^2}{M_W^2} \sim 3 \times 10^{-4}$. This mechanism of suppressing flavor (strangeness) changing neutral current is called GIM mechanism. Hence, the amplitude behaves as

$$A(K_L^0 \to \mu^+\mu^-) \propto \frac{g^4 \sin\theta_c \cos\theta_c}{M_W^2} \cdot \frac{m_c^2 - m_u^2}{M_W^2} \approx G_F^2 \sin\theta_c \cos\theta_c (m_c^2 - m_u^2). \tag{4.176}$$

Therefore, if $m_c = m_u$, $K_L^0 \to \mu^+\mu^-$ decay is strictly forbidden. Gaillard and Lee (Gaillard and Lee, 1974) predicted the mass of the introduced c quark as the value to reproduce, via the above formula, the experimental value of $Br(K_L^0 \to \mu^+\mu^-)$, before the discovery of the c quark at the collider experiments. The predicted value was quite consistent with the value ($m_c \approx 1.5 GeV$) determined from the spectroscopy of charmonium. More detailed discussion of these rare processes in the full six quark (3 generation) Kobayashi–Maskawa model will be given in Chapter 9. It is remarkable that the GIM mechanism was proposed before discovery of J/ψ in 1974.

• **Six quark case** ($n = 3$)

Here V has 3 angles and 1 phase. Because of the existence of 1 phase, we can expect the CP violation in this case. The extension of the model to the 3 generation scheme, in order to accommodate the observed CP violation in K_L decay, was first proposed by Kobayashi and Maskawa (Kobayashi and Maskawa, 1973). They generalized the Cabibbo mixing matrix, (4.167), to the six quark case of 3 generations and thus the proposed 3×3 mixing matrix V is called Cabibbo–Kobayashi–Maskawa(CKM) matrix. It is a remarkable fact that the proposal by Kobayashi and Maskawa was made before the discovery of the c quark in the 2nd generation! The GWS model extended to the Kobayashi-Maskawa 3 generation model embodies the standard model of electroweak interaction nowadays. In this case the charged weak current interaction becomes

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}}(\bar{u}\ \bar{c}\ \bar{t})\gamma^\mu \frac{1-\gamma_5}{2} V \begin{pmatrix} d \\ s \\ b \end{pmatrix} W_\mu^+ + h.c. \tag{4.177}$$

There are several pamaretrizations of the CKM matrix V, which are physically equivalent. Using 3 flavor (generation) mixing angles, θ_1, θ_2, θ_3, and a CP violating phase, δ, one example is given *à la* Kobayashi and Maskawa by

$$V = R_1(\theta_2)R_3(\theta_1)C(0,0,\delta)R_1(\theta_3), \tag{4.178}$$

where R_i is a rotation matrix around the axis i

$$R_1(\theta_i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_i & s_i \\ 0 & -s_i & c_i \end{pmatrix}, \quad R_3(\theta_i) = \begin{pmatrix} c_i & s_i & 0 \\ -s_i & c_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.179}$$

with $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$ and

$$C(0,0,\delta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix}. \tag{4.180}$$

V is explicitly written as

$$V = \begin{pmatrix} c_1 & s_1c_3 & s_1s_3 \\ -s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\ s_1s_2 & -c_1s_2c_3 - c_2s_3e^{i\delta} & -c_1s_2s_3 + c_2c_3e^{i\delta} \end{pmatrix}, \tag{4.181}$$

where the values of the 3 angles and the phase are not predictable within the framework of GWS model; they must be extracted from experiments. A phase δ

cannot be removed by the redefinition of quark fields and leads to the CP violation of the processes including δ. It is noted that the phase δ disappears if any of θ_1, θ_2, θ_3 vanishes; If $\theta_3 = 0$, $R_1(\theta_3) = 1$. Then, the phase δ can be absorbed into the newly defined fields $D' = C(0,0,\delta)D$, where $D = \begin{pmatrix} d \\ s \\ b \end{pmatrix}$, i.e. by "re-phasing" of the quark fields. For the case of $\theta_2 = 0$, using the relation $R_3(\theta_1)C(0,0,\delta) = C(0,0,\delta)R_3(\theta_1)$, the phase δ can be also removed by redefining the new field $U' = C(0,0,-\delta)U$ with $U = \begin{pmatrix} u \\ c \\ t \end{pmatrix}$. For $\theta_1 = 0$, we have $V = R_1(\theta_2)C(0,0,\delta)R_1(\theta_3)$ which leads to

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}, \tag{4.182}$$

where the 2×2 submatrix in V can be made real by suitable re-phasing, just as in the case of $n = 2$.

Another parameterization of V is the so-called "standard" parameterization which is taken by the Particle Data Group (Particle Data Group, 2002) and is characterized in terms of 3 angles θ_{12}, θ_{23}, θ_{13} and a phase δ_{13} as

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}, \tag{4.183}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ $(i, j = 1, 2, 3)$.

A convenient way to approximate V has been proposed by Wolfenstein (Wolfenstein, 1983); it is a parameterization by expanding each element of V in the powers of the sine of Cabibbo angle, $\lambda \equiv s_{12} \simeq 0.22$. In the approximation up to the order of λ^3, it is written as

$$V = \begin{pmatrix} 1 - \lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \lambda^2/2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix}, \tag{4.184}$$

where A, ρ and η are real parameters of the order one, and η corresponds to the CP violating phase. This is often used in phenomenological analysis because this parameterization shows clearly the hierarchical structure of the CKM matrix.

4.6.2.2 *The case of neutral current interaction*

Let us next consider the situation of neutral current interactions,

$$\mathcal{L}_{NC}^{(Q)} = eJ_\mu^{em}A^\mu + \frac{g}{\cos\theta_W}(J_\mu^3 - \sin^2\theta_W J_\mu^{em})Z^\mu, \tag{4.185}$$

where

$$\begin{aligned} J_\mu^{em} &= \frac{2}{3}\bar{U}\gamma_\mu U - \frac{1}{3}\bar{D}\gamma_\mu D, \\ J_\mu^3 &= \frac{1}{2}(\bar{U}_L\gamma_\mu U_L - \bar{D}_L\gamma_\mu D_L). \end{aligned} \tag{4.186}$$

These currents remain flavor diagonal, even in the base of mass eigenstates, because of the unitarity of $\mathcal{U}_U$, $\mathcal{U}_D$, $\mathcal{V}_U$ and $\mathcal{V}_D$. For example, one can easily see

$$\begin{aligned} \bar{U}\gamma_\mu U &= \bar{U}_L\gamma_\mu U_L + \bar{U}_R\gamma_\mu U_R, \\ &= \bar{U}'_L\mathcal{U}_U^\dagger\gamma_\mu\mathcal{U}_U U'_L + \bar{U}'_R\mathcal{V}_U^\dagger\gamma_\mu\mathcal{V}_U U'_R, \\ &= \bar{U}'_L\gamma_\mu U'_L + \bar{U}'_R\gamma_\mu U'_R, \\ &= U'\gamma_\mu U'. \end{aligned} \tag{4.187}$$

Therefore, we have no flavor changing neutral current at the tree level. The kinetic terms of quarks are also unchanged when we move to the mass eigenstates. Thus, we have no CP violation solely due to the weak neutral current and electromagnetic interactions.

What about Higgs boson couplings to quarks? From (4.133), (4.140) and (4.141), one can extract the Higgs–quark interactions in the unitary gauge as

$$\mathcal{L}_{Hqq} = -\frac{H}{v}(m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s + m_b\bar{b}b + m_c\bar{c}c + m_t\bar{t}t), \tag{4.188}$$

which is also flavor diagonal. Then, the Higgs couplings to quarks in the standard model with one Higgs doublet do not generate CP violation. It should be noted that this is not the case in the models with multi-Higgs, such as the models of spontaneous CP violation discussed by Lee and also by Weinberg (Lee 1974; Weinberg 1976).

4.7 Anomalies

Renormalizability is an essential principle for the theory to be meaningful at the quantum (loop) level. It is realized by sophisticated cancellation mechanism implied by gauge invariance, which is related to the current conservation called Ward–Takahashi identities (Ward, 1950; Takahashi, 1957). However, in field theories there can be the case that a conservation law holding at the tree level is violated at the quantum level by the contributions of loop diagrams. This is known as anomalies. A typical example is the triangle anomaly which is offen called the Adler–Bell–Jackiw(ABJ) anomaly or the γ_5 or axial anomaly (Adler, 1969; Bell and Jackiw, 1969; Bardeen, 1969). The "gauge anomaly", the anomaly of local gauge symmetry, is originated from the (linear) divergence of a fermion loop diagram of triangle shape with 2 vector and 1 axial vector currents at three vertices, as shown in Fig. 4.4. If this anomaly remains, the renormalizability of the theory is spoilt

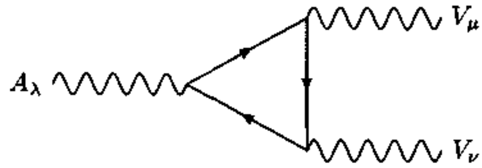


Fig. 4.4

and the model becomes meaningless at the quantum level (Georgi and Glashow, 1974; Gross and Jackiw, 1972; Bouchiat, Iliopoulos and Meyer, 1972). In general, anomaly cancellation is not trivial for the models with parity violation, containing both vector and axial vector currents, in contrast to the case of e.g. QED where only vector current exists. Thus, it is very interesting to examine if the anomaly cancellation works for the GWS model which has both vector and axial vector currents.

Let us consider, in general, the Lagrangian describing an interaction of left-handed and right-handed fermions with gauge bosons A^i_μ,

$$\mathcal{L} = \bar{\psi}_L\gamma^\mu(\partial_\mu - igT^i_L A^i_\mu)\psi_L + \bar{\psi}_R\gamma^\mu(\partial_\mu - igT^i_R A^i_\mu)\psi_R, \tag{4.189}$$

where the summation over i is implied. ψ_L and ψ_R are column vectors composed of left-handed and right-handed fermions, belonging to some representations of the gauge group, and T^i_L and T^i_R are the corresponding generators of the gauge group G satisfying

$$\left[T^i_{L,R}, T^j_{L,R}\right] = if_{ijk}T^k_{L,R}, \tag{4.190}$$

for each of T^i_L and T^i_R, where f_{ijk} is the structure constant of G. Then, the currents coupled to gauge bosons $A^{i\mu}$ are given by

$$\begin{aligned} J^i_\mu &= \bar{\psi}_L\gamma_\mu T^i_L\psi_L + \bar{\psi}_R\gamma_\mu T^i_R\psi_R, \\ &= \frac{1}{2}\bar{\psi}\gamma_\mu(1-\gamma_5)T^i_L\psi + \frac{1}{2}\bar{\psi}\gamma_\mu(1+\gamma_5)T^i_R\psi. \end{aligned} \tag{4.191}$$

Now it is known that the triangle anomaly associated with the triangle diagram with external gauge bosons A^i, A^j, A^k is proportional to

$$A^{ijk} = A^{ijk}_R - A^{ijk}_L, \tag{4.192}$$

where the contributions of the right-handed and left-handed fermions are given by

$$A^{ijk}_{R,L} = \mathrm{Tr}\left(\{T^i_{R,L}, T^j_{R,L}\}T^k_{R,L}\right), \tag{4.193}$$

and do not depend on the fermion masses. The anti-commutator in the above equation comes from the exchange of A^i and A^j gauge bosons, which are assumed to couple to the vector currents at the two vertices. Then, the condition for being anomaly free is

$$\mathrm{Tr}\left(\{T_R^i, T_R^j\}T_R^k\right) - \mathrm{Tr}\left(\{T_L^i, T_L^j\}T_L^k\right) = 0. \tag{4.194}$$

For the vector ($T_L^i = T_R^i$) or vectorlike ($T_L^i = U^{-1}T_R^iU$ with a unitary matrix U) gauge theories*, the anomaly free condition (4.194) is automatically satisfied. Typical examples of the vector theories are QED and QCD, both conserving parity symmetry. Another possibility to satisfy (4.194) is the case

$$A_R^{ijk} = A_L^{ijk} = 0. \tag{4.195}$$

This is realized, for example, for the gauge group of $SU(2)$;

$$A^{ijk} = \mathrm{Tr}\left(\{\frac{\tau^i}{2}, \frac{\tau^j}{2}\}\frac{\tau^k}{2}\right) = 0 \tag{4.196}$$

due to $\{\tau^i, \tau^j\} = 2\delta_{ij}$. This property essentially comes from the fact that the representation of $SU(2)$ is "real", and therefore the contributions of a Weyl fermion ψ and $(\psi)^c$, with c standing for the charge conjugation, should be the same, while their contributions to the anomaly should have opposite signs, as their chiralities are different.

Then, how about the GWS model? In the GWS model with only one family of leptons, matter fields are the left-handed doublet $(\nu_{eL}, e_L^-)^t$ and a right-handed singlet e_R^- and the generators for the gauge group are given by

$$T_L^i = \frac{\tau^i}{2}, \ \frac{Y_L}{2}; \qquad T_R^i = \frac{Y_R}{2}, \tag{4.197}$$

It is evident that $\mathrm{Tr}\left(\{\frac{\tau^i}{2}, \frac{\tau^j}{2}\}\frac{\tau^k}{2}\right) = 0$ as shown above. Then, for the model to be anomaly free we must impose

$$\mathrm{Tr}\left(\{\frac{\tau^i}{2}, \frac{\tau^j}{2}\}Y_L\right) = \frac{1}{2}\delta_{ij}\mathrm{Tr}Y_L = \frac{1}{2}\delta_{ij}\mathrm{Tr}(2Q - \tau^3) = \delta_{ij}\mathrm{Tr}Q = 0, \tag{4.198}$$

and

$$\mathrm{Tr}Y_L^3 - \mathrm{Tr}T_R^3 = 0. \tag{4.199}$$

(Note that another possible anomaly associated with a diagram with currents accompanied by single τ^i and two weak-hypercharge generators Y simply vanishes, thanks to the relation $\mathrm{Tr}\tau^i = 0$.) Here for right-handed singlets, $Q = \frac{Y_R}{2}$ and thus we have

$$\mathrm{Tr}Y_R^3 = 8\mathrm{Tr}Q^3 \propto \mathrm{Tr}Q, \tag{4.200}$$

*Currents are written as $J_\mu^i = \bar{\psi}_L\gamma_\mu T_L^i\psi_L + \bar{\psi}_R\gamma_\mu T_R^i\psi_R = \bar{\psi}_L\gamma_\mu T_L^i\psi_L + \bar{\psi}_R\gamma_\mu UT_L^iU^{-1}\psi_R = \bar{\psi}\gamma_\mu T^i\psi$, where $\psi = \psi_L + U^{-1}\psi_R$ and $T^i = T_L^i$.

since $Q^3 = Q$ for leptons, and for left-handed doublets, we see

$$\begin{aligned}
\mathrm{Tr}Y_L^3 &= \mathrm{Tr}(2Q - \tau^3)^3 \\
&= 8\mathrm{Tr}Q^3 + 6\mathrm{Tr}\left(Q(\tau^3)^2\right) - 6\mathrm{Tr}(Q^2\tau^3) - \mathrm{Tr}\left((\tau^3)^3\right) \\
&\propto \mathrm{Tr}Q,
\end{aligned} \tag{4.201}$$

where we have used relations $(\tau^3)^2 = 1$, $\mathrm{Tr}(\tau^3)^3 = 0$ and $\mathrm{Tr}(Q^2\tau^3) = \mathrm{Tr}Q$ for leptons. After all, for the GWS model to be anomaly free, it is necessary to have

$$\mathrm{Tr}Q = 0. \tag{4.202}$$

This condition is not satisfied in the GWS model only with leptons and should be satisfied by extending the model so that it includes quarks with 3 color degree of freedom. For example, for the 1st generation model of leptons and quarks with N_c color degree of freedom

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, e_R; \quad \begin{pmatrix} u \\ d \end{pmatrix}_L, u_R, d_R, \tag{4.203}$$

the condition of anomaly cancellation

$$\mathrm{Tr}Q = 0 - 1 + N_c\left(\frac{2}{3} - \frac{1}{3}\right) = 0 \tag{4.204}$$

can be satisfied only for $N_c = 3$, where 0, -1, $\frac{2}{3}$ and $-\frac{1}{3}$ represent the charges of ν_e, e, u and d, respectively. (Though the statement that (4.199) is equivalent to (4.202) was made for the pure leptonic model, it turns out to hold even after the inclusion of the quarks.) The lepton anomaly can be cancelled by the quark anomaly only when the color degree of freedom is 3. Therefore, the standard model with $G = SU(3)_c \times SU(2)_L \times U(1)_Y$ suggests that leptons and quarks are closely related to each other. The real unification of quarks and leptons in the multiplets of gauge group is realized in the "Grand Unified Theories (GUT)" to unify strong and electroweak interactions, such as $SU(5)$ or $SO(10)$ gauge theories.

Problems

4.1 Prove the commutation relations $[Q - T^3, T^i] = 0$ for $i = 1, 2, 3$, where Q is the electric charge operator defined by (4.18) and T^i are weak isospin operators defined by (4.13).

4.2 Show that the Yukawa interaction defined by (4.33) is invariant under $SU(2)_L \times U(1)_Y$ gauge transformation.

4.3 Show that in the GWS model, the electric charge operator Q is not a broken generator, though weak isospin operators T^i violate the symmetry of the vacuum given by (4.36).

4.4 Derive (4.68).

4.5 Show that arbitrary $n \times n$ complex matrices can be diagonalized by bi-unitary tarnsformations.

4.6 The lowest order decay amplitude of $W \to \ell\bar{\nu}_\ell$ ($\ell = e, \mu, \tau$) is given by

$$M = \frac{g}{2\sqrt{2}}\bar{u}(p)\gamma_\mu(1-\gamma_5)v(q)\epsilon^\mu(k,\lambda),$$

where $\epsilon^\mu(k,\lambda)$ is the polarization vector of the W boson with helicity λ. Then, neglecting the lepton masses, calculate the decay width for this decay which is given as

$$\Gamma(W \to \ell\bar{\nu}_\ell) = \frac{G_F M_W^3}{2\sqrt{6}\pi}.$$

4.7 The interaction between Z^0 and a fermion f is given by

$$\mathcal{L}_{NC}^Z = \frac{g}{\cos\theta_W} J_\mu^Z Z^\mu,$$

where J_μ^Z is given by (4.97).
(1)Neglecting the fermion mass, calculate the decay width of $Z^0 \to f\bar{f}$ in the tree level, which is given as

$$\Gamma(Z^0 \to f\bar{f}) = \frac{2G_F M_Z^3}{3\sqrt{2}\pi}\left((C_V^f)^2 + (C_A^f)^2\right),$$

where C_V^f, C_A^f are given in (4.98)~(4.101).
(2)Since the top quark is heavier than Z^0, Z^0 cannot decay into a top quark pair. Then, assuming $\sin^2\theta_W \simeq 0.22$, estimate the branching ratio of $Z^0 \to e^+e^-$.

Chapter 5

QUANTUM CHROMODYNAMICS

Quantum chromodynamics (QCD) is the theory of strong interactions which describes the dynamics of quarks and gluons. QCD is the non-Abelian gauge theory with $SU(3)$ color symmetry. In 1930s, the idea of strong interactions was first introduced by Yukawa to explain new strong forces those days, called nuclear forces, between nucleons mediated by pions. Before long, many hadrons, in addition to nucleons and pions, were discovered and it was revealed that the interactions among those hadrons were very complicated and did not seem to be fundamental.

Now, we know that all hadrons are composite particles of quarks which were first introduced by Gell-Mann and Zweig in 1964 (Gell-Mann, 1964; Zweig, 1964) to explain the spectroscopy of hadrons those days. The quarks were established as the fundamental constituents of hadrons in the development of the quark model in 1960s and 1970s; baryons are composed of three quarks qqq and mesons of a quark q and an antiquark $\bar{q}$. The quarks are bound inside hadrons through strong interactions mediated by gluons with the coupling of color charges $g_s\frac{\lambda^i}{2}$, where λ^i $(i = 1, 2, \cdots, 8)$ are the 3×3 Gell-Mann matrices as explicitly presented in Appendix (C.4). The situation in hadrons is quite similar to the case of positronium where the electromagnetic Coulomb forces between a positron and an electron are mediated by photons with the coupling of electric charge e. It is well-known that interactions between electrons(or positrons) and photons are beautifully bescribed by quantum elecrodynamics(QED), i.e. the gauge theory of $U(1)$ symmetry called $U(1)_{em}$, which is invariant under local Abelian $U(1)$ phase transformation. QCD which describes the strong interactions between quarks and gluons is also the gauge theory but this time it is not an Abelian theory but a non-Abelian theory of $SU(3)$ color symmetry, described as $SU(3)_c$, which is invariant under local non-Abelian $SU(3)$ transformation in color space. Notice that the color being in 3 species as R(red), B(blue) and G(green), is unrelated to another attribute of quarks, i.e. the flavor, which is in 6 species (u, d, s, c, b, t). In contrast to the flavor symmetry like the $SU(3)_f$ symmetry among u, d and s quarks which is rather largely broken, the color $SU(3)_c$ symmetry is unbroken and exact as well as the $U(1)_{em}$ symmetry in QED. Therefore, just like photons in QED, gluons as gauge bosons mediating strong interactions are also massless.

Since there exist no colored hadrons in Nature, we assume that all observed hadrons must be colorless, i.e. in the color singlet states. This seems to be quite similar to the case of isospin states for two-nucleon systems. As is well known, though we can expect, at a glance, to have two different isospin states $I = 1$(isotriplet) and $I = 0$(isosinglet) for the two-nucleon systems composed of a proton and a neutron, actually only $I = 0$ state is allowed to exist as a deuteron. However, one should notice that there is a big difference between hadrons and deuteron; though in the case of deuteron, a proton and a neutron which are constituents of deuteron exist separately in Nature, colored quarks which are constituents of hadrons do not exist separately in Nature but are confined inside hadrons as colorless states. This is the so-called confinement problem which should be explained in QCD.

In this chapter, after describing the evidence of the color degree of freedom, we discuss the coupling strength of color forces between two quarks and also between a quark and an antiquark. Then, we discuss the running coupling constants in QED and QCD cases and discuss the difference between QED and QCD.

5.1 Evidence for colors

There are several evidences for existence of the color degree of freedom, which is actually in 3 species. The color was first introduced to solve the difficulty in the relation of spin and statistics in the baryon spectroscopy. In the quark model, baryons are made of three spin $\frac{1}{2}$ quarks. Without color space, the wavefunction of a baryon is described by the product of space-, spin- and flavor-wavefunctions. Then, consider the $\Delta^{++}(1232)$ with spin $\frac{3}{2}$. It is the ground state of the uuu bound system and hence its wavefunction is totally symmeric under interchange of any u quark pairs in space-, spin- and flavor-wavefunctions. However, since the $\Delta^{++}(1232)$ is a fermion with spin $\frac{3}{2}$, its total wavefunction must be antisymmetric under interchange of any u quark pairs. This difficulty can be solved by introducing an antisymmetric color space wavefunction, where the color space is a new internal space.

A more direct evidence for color to be in 3 species comes from the experimental data on e^+e^- annihilations at high energies. Based on the quark parton model, the ratio

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} \tag{5.1}$$

is predicted to be $R = \sum_q e_q^2$, where e_q is the electric charge of the quark q in quark pairs produced in the e^+e^- annihilation. Beyond the s quark pair production threshold but for not so high energies, only u, d and s quarks contribute to this

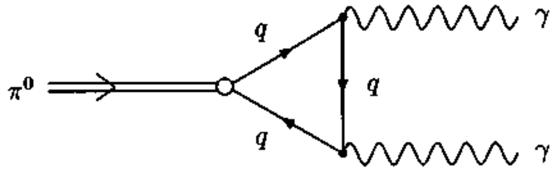


Fig. 5.1

ratio R and yield

$$\begin{aligned} R &= e_u^2 + e_d^2 + e_s^2 = \frac{2}{3}, \quad \text{(without color)} \\ R &= 2. \quad \text{(with 3 colors)} \end{aligned} \tag{5.2}$$

Similarly, for higher energies beyond the c quark pair production threshold, R becomes

$$\begin{aligned} R &= e_u^2 + e_d^2 + e_s^2 + e_c^2 = \frac{10}{9}, \quad \text{(without color)} \\ R &= \frac{10}{3}, \quad \text{(with 3 colors)} \end{aligned} \tag{5.3}$$

and for more higher energies beyond the b quark pair production, it becomes

$$\begin{aligned} R &= e_u^2 + e_d^2 + e_s^2 + e_c^2 + e_b^2 = \frac{11}{9}, \quad \text{(without color)} \\ R &= \frac{11}{3}. \quad \text{(with 3 colors)} \end{aligned} \tag{5.4}$$

Experimental data prefer the 3 colors for any cases.

Another important observation suggesting the color number to be 3 is the decay rate of $\pi^0 \to \gamma\gamma$. The process proceeds through the coupling of the pion to a quark loop as shown in Fig. 5.1, in which the u and d quark loops contribute to this decay kinematically. The calculated result is given by

$$\Gamma(\pi^0 \to \gamma\gamma) = N_c^2 (e_u^2 - e_d^2)^2 \left(\frac{\alpha}{\pi}\right)^2 \frac{m_\pi^3}{32\pi f_\pi}, \tag{5.5}$$

where N_c is the number of colors and $f_\pi \simeq 130$MeV is the pion decay constant. Then, we can predict the rate to be

$$\Gamma(\pi^0 \to \gamma\gamma) = \begin{cases} 0.86 \text{ eV}, & \text{(without color } (N_c = 1)) \\ 7.65 \text{ eV}. & \text{(with 3 colors } (N_c = 3)) \end{cases} \tag{5.6}$$

The measured rate is (7.74 ± 0.55) eV which agrees with $N_c = 3$.

Further example is the branching ratio of the τ lepton, $Br = \frac{\Gamma(\tau^- \to e^- \bar{\nu}_e \nu_\tau)}{\Gamma(\tau^- \to all)}$. A τ^- lepton decays into kinematically allowed lepton pairs, $(e^-\ \bar{\nu}_e)$, $(\mu^-\ \bar{\nu}_\mu)$, and quark pairs, $(d\ \bar{u})$, $(s\ \bar{u})$, through the diagram shown in Fig. 5.2. Among these channels, the contribution of $(s\ \bar{u})$ is very small because it is a Cabibbo-suppressed

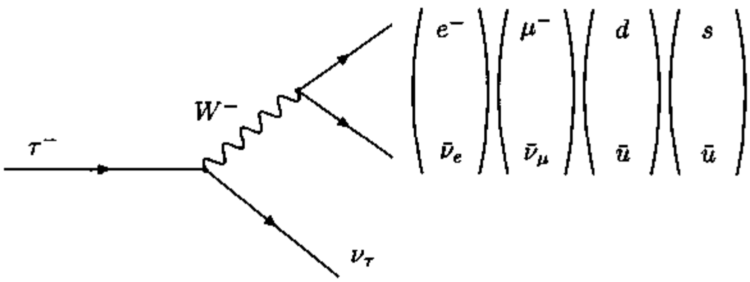


Fig. 5.2

process and hence can be neglected. Since the coupling is same for all these channels, we can simply predict the above branching ratio as

$$\begin{aligned} Br &= \frac{1}{3}, \quad \text{(without color)} \\ Br &= \frac{1}{5}. \quad \text{(with 3 colors)} \end{aligned} \tag{5.7}$$

The experimental data is $Br = (17.84 \pm 0.06)\%$ which is again in agreement with 3 colors.

5.2 QCD Lagrangian and the strength of color forces

The QCD Lagrangian describing the interaction between quarks q and gluons A_μ^i ($i = 1, 2, \cdots, 8$) is given by

$$\mathcal{L} = \bar{q}(i\not{D} - m)q - \frac{1}{4}F_{\mu\nu}^i F^{i\mu\nu} \tag{5.8}$$

where the summation over i ($i = 1, 2, \cdots, 8$) is implied. The quark field q is given in both the Dirac space and the color space with three color components as $q = \begin{pmatrix} q^R \\ q^G \\ q^B \end{pmatrix}$, where the superscripts, R, G and B, denote red, green and brue, respectively. The covariant derivative is given by

$$D_\mu = \partial_\mu - ig_s \frac{\lambda^i}{2} A_\mu^i, \tag{5.9}$$

where g_s is the strong coupling constant and λ^i are the 3×3 Gell-Mann matrices presented in Appendix (C.4). The summation over i ($i = 1, 2, \cdots, 8$) is implied. F_μ^i are the field strength tensor for gluon fields A_μ^i and given by

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + g_s f_{ijk} A_\mu^j A_\nu^k, \tag{5.10}$$

where f_{ijk} are the structure constant of the color $SU(3)_c$ group.

Here we are interested in the strength of color interactions between two quarks and also between a quark and an antiquark. Using the explicit expression of λ^i presented in (C.4), the interaction between quarks and gluons given in (5.8) is written as

$$\begin{aligned}
\mathcal{L}_{int} &= g_s\bar{q}\gamma_\mu\frac{\lambda^i}{2}qA^i_\mu \\
&= \frac{g_s}{\sqrt{2}}\{\bar{q}^R\gamma^\mu q^G G^1_\mu + \bar{q}^G\gamma^\mu q^R G^2_\mu \\
&+ \frac{1}{\sqrt{2}}(\bar{q}^R\gamma^\mu q^R - \bar{q}^G\gamma^\mu q^G)G^3_\mu + \bar{q}^R\gamma^\mu q^B G^4_\mu + \bar{q}^B\gamma^\mu q^R G^5_\mu \\
&+ \bar{q}^G\gamma^\mu q^B G^6_\mu + \bar{q}^B\gamma^\mu q^G G^7_\mu \\
&+ \frac{1}{\sqrt{6}}(\bar{q}^R\gamma^\mu q^R + \bar{q}^G\gamma^\mu q^G - 2\bar{q}^B\gamma^\mu q^B)G^8_\mu\},
\end{aligned} \tag{5.11}$$

where

$$\begin{array}{lll}
G^1_\mu = (A^1_\mu - iA^2_\mu)/\sqrt{2}, & G^2_\mu = (A^1_\mu + iA^2_\mu)/\sqrt{2}, & G^3_\mu = A^3_\mu \\
G^4_\mu = (A^4_\mu - iA^5_\mu)/\sqrt{2}, & G^5_\mu = (A^4_\mu + iA^5_\mu)/\sqrt{2}, & \\
G^6_\mu = (A^6_\mu - iA^7_\mu)/\sqrt{2}, & G^7_\mu = (A^6_\mu + iA^7_\mu)/\sqrt{2}, & G^8_\mu = A^8_\mu
\end{array} \tag{5.12}$$

represent redefined gluon fields. As can been seen here, the gluons come in 8 different color combinations, $\bar{R}G$, $\bar{G}R$, $\frac{1}{\sqrt{2}}(\bar{R}R - \bar{G}G)$, $\bar{R}B$, $\bar{B}R$, $\bar{G}B$, $\bar{B}G$, $\frac{1}{\sqrt{6}}(\bar{R}R + \bar{G}G - 2\bar{B}B)$. This means that the gluons belong to a color $SU(3)_c$ octet. For example, G^1_μ changes the color of the quark from G(green) to R(red). Another combination, i.e. the color singlet $\frac{1}{\sqrt{3}}(\bar{R}R + \bar{G}G + \bar{B}B)$, does not mediate color charge.

Now we are concerned with the interactions between two quarks. First let us consider the interaction between R(red) and R(red) which arises from 2 diagrams due to G^3_μ and G^8_μ exchange as shown in Fig. 5.3(a) and (b). Then, the coupling of this interaction is obtained from (5.11) as

$$\left(\frac{1}{\sqrt{2}}\frac{g_s}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{6}}\frac{g_s}{\sqrt{2}}\right)^2 = \frac{2}{3}\left(\frac{g_s}{\sqrt{2}}\right)^2. \tag{5.13}$$

The interaction between B(green) and B(green) arises only from G^8_μ exchange as shown in Fig. 5.3(c) and the coupling becomes

$$\left(-\frac{2}{\sqrt{6}}\frac{g_s}{\sqrt{2}}\right)^2 = \frac{2}{3}\left(\frac{g_s}{\sqrt{2}}\right)^2. \tag{5.14}$$

The interaction between R(red) and G(green) arises from 3 diagrams shown in Fig. 5.3(d), (e) and (f) and the coupling becomes

$$\left(\frac{1}{\sqrt{6}}\frac{g_s}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\frac{g_s}{\sqrt{2}}\right)\cdot\left(-\frac{1}{\sqrt{2}}\frac{g_s}{\sqrt{2}}\right) + \left(\frac{g_s}{\sqrt{2}}\right)^2 = \frac{2}{3}\left(\frac{g_s}{\sqrt{2}}\right)^2. \tag{5.15}$$

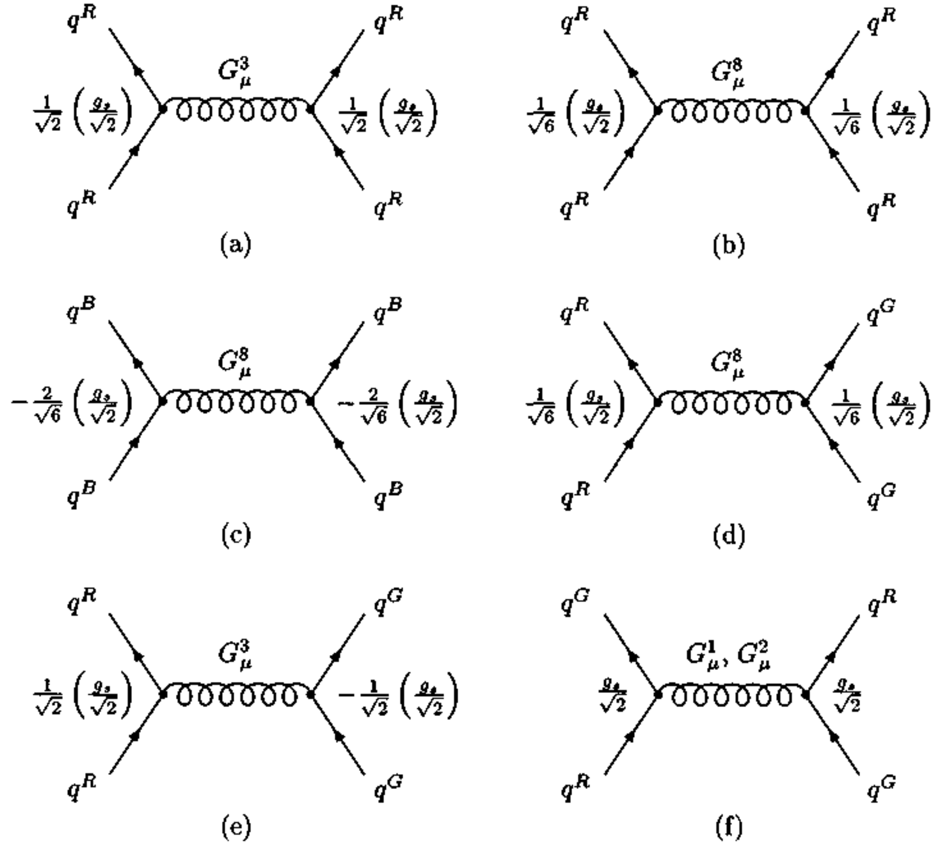


Fig. 5.3

As seen in the above examples, all color interactions between two quarks (color–color interactions) are equal as expected from exact color symmetry. In other words, two quark interactions are blind to the difference of colors.

What about the interactions between a quark and an antiquark (color–anti-color interactions) in the color-singlet state, $\frac{1}{\sqrt{3}}(\bar{R}R + \bar{G}G + \bar{B}B)$, i.e. between q and $\bar{q}$ in a meson? Here, since each color is in equal weight in the color-singlet state of a meson, it is enough to consider, for example, the case of B(blue) and $\bar{B}$(anti-blue) whose interaction arises from 3 diagrams shown in Fig. 5.4(a), (b) and (c). (The situation is same for between R(red) and R(red) and also between G(green) and G(green).) Then, the coupling of this interaction becomes

$$\left(\frac{2}{\sqrt{6}}\frac{g_s}{\sqrt{2}}\right)\cdot\left(-\frac{2}{\sqrt{6}}\frac{g_s}{\sqrt{2}}\right)+\left(\frac{g_s}{\sqrt{2}}\right)\cdot\left(-\frac{g_s}{\sqrt{2}}\right)+\left(\frac{g_s}{\sqrt{2}}\right)\cdot\left(-\frac{g_s}{\sqrt{2}}\right)=-\frac{4}{3}\left(\frac{g_s}{\sqrt{2}}\right)^2. \tag{5.16}$$

Here we should notice that the coupling between antiquark and gluon is in the opposite sign to the one between quark and gluon because of vector nature of gluons,

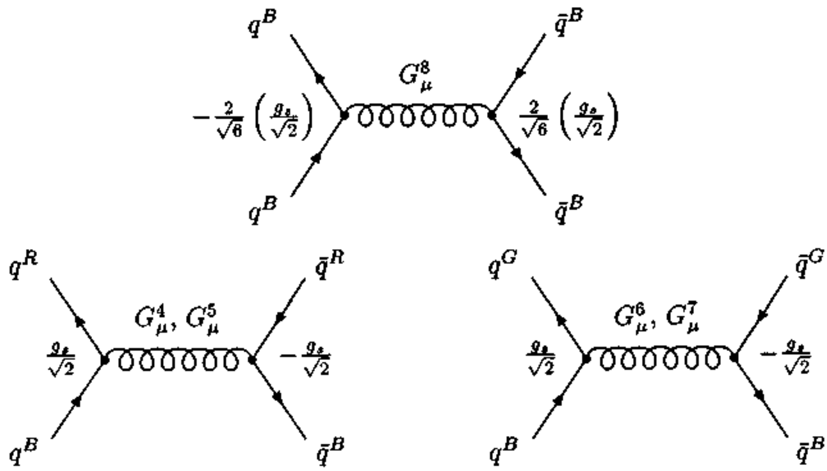


Fig. 5.4

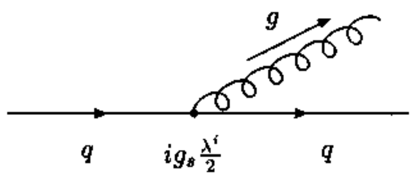


Fig. 5.5

just as in QED mediated by a photon where the charge of positron (anti-particle of electron) is opposite to the electron charge. We have the same contribution of $\bar{R}R$ and $\bar{G}G$ as the one of $\bar{B}B$ obtained in (5.16). However, when we consider the color–singlet state of mesons, each of initial and final states in Fig. 5.4 has a factor $\frac{1}{\sqrt{3}}$. Then, taking into account of these factors as $3 \times \left(\frac{1}{\sqrt{3}}\right)^2 = 1$, the coupling for $q\bar{q}$ interactions in a meson becomes the same as given in (5.16).

The similarity of QCD and QED is in many cases useful in calculating processes in the tree approximation. Let us take one such example, a process, $q \to q + g$ (Fig. 5.5). The matrix element of this process is given as

$$M = g_s \bar{u}^b \gamma^\mu \left(\frac{\lambda^i}{2}\right)_{ba} u^a A^i_\mu, \tag{5.17}$$

which is essentially same for the QED process, $q \to q + \gamma$ except for the coupling constant and the color matrix factor. a and b denote the color indices. Squaring the above matrix element, one can extract the color-dependent factor as

$$\frac{1}{3} g_s^2 \mathrm{Tr}\left[\frac{\lambda^i}{2}\frac{\lambda^i}{2}\right] = \frac{4}{3} g_s^2, \tag{5.18}$$

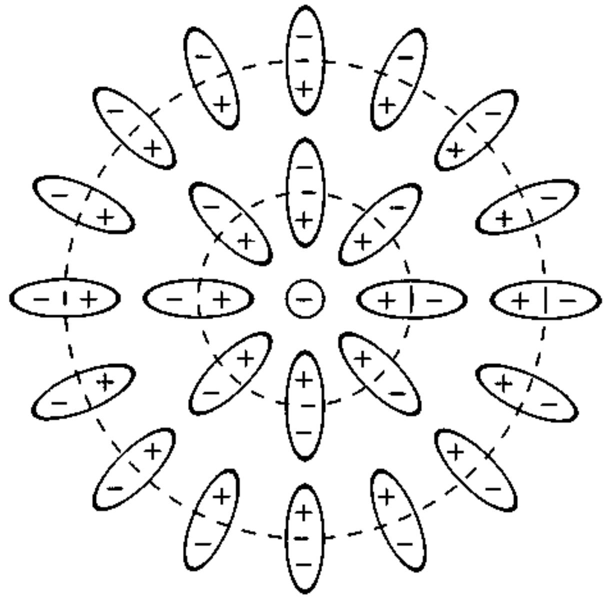


Fig. 5.6

using the relation $\mathrm{Tr}\left[\frac{\lambda^i}{2}\frac{\lambda^i}{2}\right] = 4$, where the summation over i ($i = 1, 2, \cdots, 8$) is implied. In (5.18), a factor $\frac{1}{3}$ is due to averaging over the initial quark's color states (=3). Then, the transformation from QED process to QCD process can be done simply by replacing α by $\frac{4}{3}\alpha_s$ in the transition probability for the corresponding QED process.

5.3 Running coupling constants

5.3.1 *Renormalized charge in QED*

In QED, an electron shows many guises; an electron virtually emits photons which produce electron–positron pairs and those pairs emit further photons and so on. Hence, an electron turns out to be surrounded by many virtual electrons and positrons and due to the Coulomb attractive force, positons become closer to the original elecrton (which we call the "bare" electron) and thus the vacuum is polarized as shown in Fig. 5.6. Because of this vacuum polarization, the effective charge (observed charge) of an electron decreases as we go away from the bare electron, that is to say, the observed charge e or the fine-structure constant, $\alpha = \frac{e^2}{4\pi}$, is not a constant but depends on the distance r from the bare charge or the momentum transfer Q^2 in scattering by a test charge. In fact, the value of $\alpha(Q^2)$ decreases with decreasing Q^2 (or increasing r) and thus the vacuum polarization leads to the charge screening as shown in Fig. 5.7.

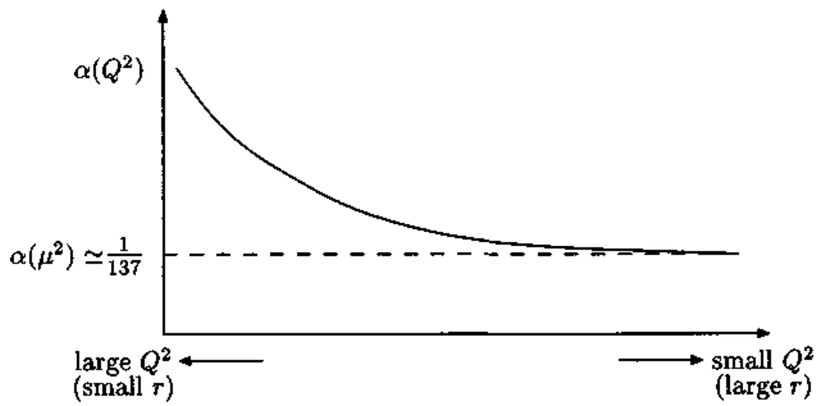


Fig. 5.7

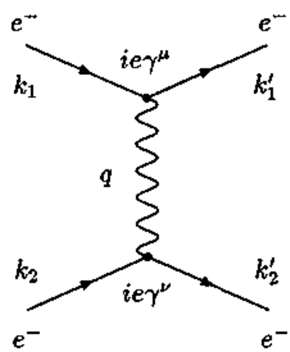


Fig. 5.8

To understnad this property in QED, let us consider the electron–electron scattering, $e^-(k_1) + e^-(k_2) \to e^-(k_1') + e^-(k_2')$. The lowest order amplitude, being in $\mathcal{O}(\alpha)$, for this process is given by (see Fig. 5.8)

$$-iM = [ie\bar{u}(k_1')\gamma^\mu u(k_1)]\frac{-ig_{\mu\nu}}{q^2}[ie\bar{u}(k_2')\gamma^\nu u(k_2)], \tag{5.19}$$

where $q = k_1 - k_1'$ is the vertual photon momentum. However, to get an exact amplitude for this scattering, we need to calculate higher order corrections containing all order of α. The first order correction to the above amplitude comes from the diagram of Fig. 5.9, in which the photon propagator contains an electron loop, and according to the standard Feynman rule, it is written as

$$-iM \;=\; (-1)[ie\bar{u}(k_1')\gamma^\mu u(k_1)]\frac{-ig_{\mu\rho}}{q^2}$$

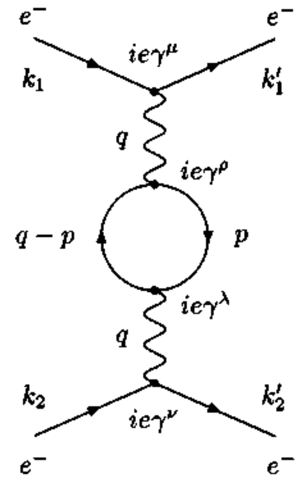


Fig. 5.9

$$\begin{aligned} &\times \quad \frac{1}{(2\pi)^4}\int d^4p \,\mathrm{Tr}\left[(ie\gamma^\rho)\frac{i(\not p+m)}{p^2-m^2}(ie\gamma^\lambda)\frac{i(\not q-\not p+m)}{(p-q)^2-m^2}\right] \\ &\times \quad \frac{-ig_{\lambda\nu}}{q^2}[ie\bar{u}(k_2')\gamma^\nu u(k_2)], \end{aligned} \tag{5.20}$$

where p is the loop momentum of the internal fermion(electron) and m denotes the electron mass. The factor (-1) is originated from the fact that this diagram contains one fermion loop. It should be noticed that the internal momentum p should be integrated from 0 to ∞ and leads to the divergence of the amplitude. (Exact calculation results in only a logarithmic divergence, though one might expect it to be quadratic because of the apparent form $\int_0^\infty pdp$ of the integral in (5.20).) This divergence can be removed in QED by the renormalization technique as discussed below. Note that though the amplitude (5.20) is of the $\mathcal{O}(\alpha^2)$, this one-loop correction to the photon propagator is of the $\mathcal{O}(\alpha)$.

Now, one might worry about additional contributions of the same $\mathcal{O}(\alpha)$ originated from other diagrams depicted in Fig. 5.10, where (a) and (b) are the self-energy terms of electrons and (c) is the vertex correction. However, to our surprise, QED resuts in exact cancellation of the sum of the contribution of (a) and (b) and the one of (c). Actually this cancellation occurs in every order of perturbation. This is the fundamental property of QED and known as the Ward–Takahashi identity. Because of this cancellation, it suffices to evaluate only the modification to the photon propagator.

The addition of (5.20) to (5.19) can be considered as modification of the photon propagator as

$$\frac{-ig_{\mu\nu}}{q^2} \quad \to \quad \frac{-ig_{\mu\nu}}{q^2}+\frac{-ig_{\mu\rho}}{q^2}I^{\rho\lambda}\frac{-ig_{\lambda\nu}}{q^2}$$

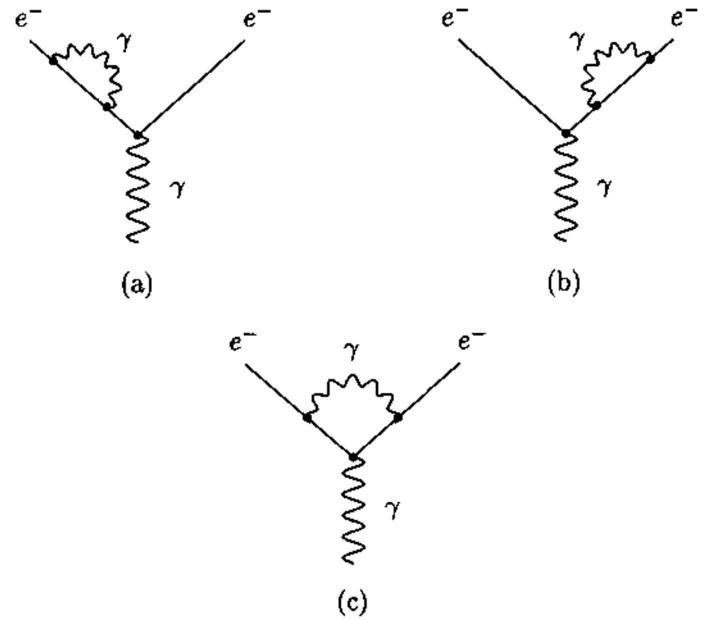


Fig. 5.10

$$= \quad \frac{-ig_{\mu\nu}}{q^2} + \frac{-i}{q^2} I_{\mu\nu} \frac{-i}{q^2}, \tag{5.21}$$

where

$$I_{\mu\nu}(q^2) = \frac{(-1)}{(2\pi)^4} \int d^4p \ \mathrm{Tr} \left[(ie\gamma_\mu) \frac{i(\not{p} + m)}{p^2 - m^2} (ie\gamma_\nu) \frac{i(\not{q} - \not{p} + m)}{(q-p)^2 - m^2} \right]. \tag{5.22}$$

After a lengthy calculation, we find that the one-loop correction to the photon propagator can be written as

$$I_{\mu\nu} = -ig_{\mu\nu} q^2 I(q^2) + \cdots \tag{5.23}$$

with

$$I(q^2) = \frac{\alpha}{3\pi} \int_{m^2}^{\infty} \frac{dp^2}{p^2} - \frac{2\alpha}{\pi} \int_0^1 dx \ x(1-x) \log \left(1 - \frac{q^2(1-x)}{m^2} \right). \tag{5.24}$$

The dots in (5.23) contain terms proportional to $q_\mu q_\nu$ and vanish when the propagator is coupled to external electron currents. Substituting (5.23) in (5.21), we obtain

$$\frac{-ig_{\mu\nu}}{q^2} \to \frac{-ig_{\mu\nu}}{q^2}(1 - I(q^2)). \tag{5.25}$$

Therefore, we find that in the $O(\alpha)$ the modification of the photon propagator can be made just by mutiplying a factor $(1 - I(q^2))$. Explicit derivations of these formulas are beyond this textbook and can be found, for example, in the books by Bjorken and Drell (Bjorken and Drell, 1964) or Jauch and Rohrlich (Jauch and Rohrlich,1976). Here, we just comment on some important points of results. $I(q^2)$ is infinite as mentioned above. But the infinity arises only from the first term of (5.24) which is logarithmically divergent and independent of q^2. One can estimate this integral by introducing the cut-off parameter Λ for the upper limit of the integral and then by approaching it to ∞. Actual calculation will be done for the two extremes, i.e. $(-q^2) \gg m^2$ and $(-q^2) \ll m^2$.

Let us first consider the case of $(-q^2) \gg m^2$. In this case, $\log\left(1 - \frac{q^2 x(1-x)}{m^2}\right) \simeq \log(\frac{-q^2}{m^2})$ and then, we can evaluate $I(q^2)$ as

$$\begin{aligned} I(q^2) &= \frac{\alpha}{3\pi}\int_{m^2}^{\Lambda^2}\frac{dp^2}{p^2} - \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\log\left(\frac{-q^2}{m^2}\right) \\ &= \frac{\alpha}{3\pi}\log\left(\frac{\Lambda^2}{m^2}\right) - \frac{\alpha}{3\pi}\log\left(\frac{-q^2}{m^2}\right) = \frac{\alpha}{3\pi}\log\left(\frac{\Lambda^2}{-q^2}\right). \end{aligned} \tag{5.26}$$

Thus, for large $(-q^2)$, the photon propagator is changed only by mutiplying a factor $(1 - \frac{\alpha}{3\pi}\log(\frac{\Lambda^2}{m^2}))$. The multiplicative factor from loop corrections of all orders in α can be also evaluated as

$$1 - \frac{\alpha}{3\pi}\log\left(\frac{\Lambda^2}{-q^2}\right) + \mathcal{O}(\alpha^2) = \frac{1}{1 + \frac{\alpha}{3\pi}\log\left(\frac{\Lambda^2}{-q^2}\right)}, \tag{5.27}$$

and thus, the full amplitude becomes

$$-iM = [ie\bar{u}(k_1')\gamma^\mu u(k_1)]\frac{-ig_{\mu\nu}}{q^2}\left(\frac{1}{1 + \frac{\alpha}{3\pi}\log\left(\frac{\Lambda^2}{-q^2}\right)}\right)[ie\bar{u}(k_2')\gamma_\mu u(k_2)], \tag{5.28}$$

which suggests that the effective charge or fine-structure constant α_{eff} is given in terms of $\alpha = \frac{e^2}{4\pi}$ by

$$\alpha_{eff} = \frac{\alpha}{1 + \frac{\alpha}{3\pi}\log\left(\frac{\Lambda^2}{-q^2}\right)}. \tag{5.29}$$

So far, we considered the value of $\alpha = \frac{e^2}{4\pi}$ to be a constant, which is the coupling at the vertex of a "bare" electron and photon. We call it as the bare or unrenormalized charge/coupling and write it as α_0 hereafter. On the other hand, the observed coupling, usually called the fine-structure constant ($\approx \frac{1}{137}$), is different from this and called the renormalized charge/coupling, which is, in practice, given by α_{eff} in (5.29) and we write it as $\alpha(Q^2)$ hereafter, where $Q^2 = -q^2$. Then, we rewrite (5.29) as

$$\alpha(Q^2) = \frac{\alpha_0}{1 + \frac{\alpha_0}{3\pi}\log\left(\frac{\Lambda^2}{Q^2}\right)}, \tag{5.30}$$

where $\alpha(Q^2)$ and α_0 represent the renormalized and bare charge/coupling, respectively.

Now, suppose that we can get the result, $\alpha(\mu^2) = \frac{1}{137}$, in experiment at some value of $Q^2 = \mu^2$. Then, using the formula (5.30) for $\alpha(\mu^2)$, one can obtain

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - \frac{\alpha(\mu^2)}{3\pi} \log\left(\frac{Q^2}{\mu^2}\right)}. \tag{5.31}$$

Notice that in (5.31) there is no dependence on the cut-off Λ and α_0. $\alpha(Q^2)$ depends only on the finite measurable quanitities. Since $\alpha(Q^2)$ depends on Q^2, $\alpha(Q^2)$ is called the "running coupling constant". (5.31) shows the charge screening, that is to say, $\alpha(Q^2)$ decreases (increases) with decreasing (increasing) Q^2. (See Fig. 5.7) Here we introduced a parameter μ with the dimension of mass to avoid the infinity of the amplitude by renormalizing the charge. μ is called the renormarization mass(or scale). Different choice of μ corresponds to different renormalization schemes but final result does not depend on the choice of μ. This is because the dependence of the amplitude on Λ is absorbed into $\alpha(\mu^2)$.

Now, let us move to the oppsite extreme of small $(-q^2)$. This limit is relevant in the Coulomb scattering of an electron by a static nucleus terget with charge Ze. In the limit $(-q^2) \to 0$, we see $\log(1 - q^2x(1-x)/m^2) \simeq -q^2x(1-x)/m^2$ and thus, (5.24) reduces to

$$I(q^2) = \frac{\alpha}{3\pi} \log\left(\frac{\Lambda^2}{m^2}\right) + \frac{\alpha}{15\pi}\frac{q^2}{m^2} \tag{5.32}$$

where we again introduce the cut-off parameter Λ for upper limit of the first term of the integral (5.24). Note that the divergent term, i.e. the first term of this result is of the same form as (5.26), just by replacing m^2 by $-q^2$.

Then, to order α^2, the matrix element for the Coulomb scattering of an electron by a charge Ze is calculated by replacing the factor $[ie\bar{u}(k_2')\gamma^\nu u(k_2)]$ in (5.20) by $-ij^\nu$ with $j^\nu = (\rho = Ze, \vec{j} = 0)$

$$\begin{aligned} -iM &= [ie\bar{u}(k_1')\gamma^\mu u(k_1)]\frac{-ig_{\mu\nu}}{q^2}\left[1 - \frac{\alpha}{3\pi}\log\left(\frac{\Lambda^2}{m^2}\right) - \frac{\alpha}{15\pi}\frac{q^2}{m^2}\right](-ij^\nu) \\ &= [ie\bar{u}(k_1')\gamma_0 u(k_1)]\frac{-i}{q^2}\left[1 - \frac{\alpha}{3\pi}\log\left(\frac{\Lambda^2}{m^2}\right) - \frac{\alpha}{15\pi}\frac{q^2}{m^2}\right](-iZe). \end{aligned} \tag{5.33}$$

When we write this result as

$$-iM = [ie_R\bar{u}(k_1')\gamma_0 u(k_1))]\frac{-i}{q^2}(-iZe_R), \tag{5.34}$$

with

$$e_R = e\left[1 - \frac{\alpha}{3\pi}\log\left(\frac{\Lambda^2}{m^2}\right)\right]^{1/2}, \tag{5.35}$$

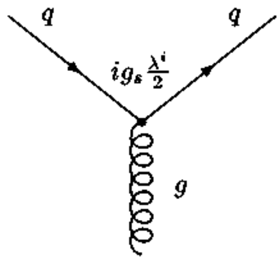


Fig. 5.11

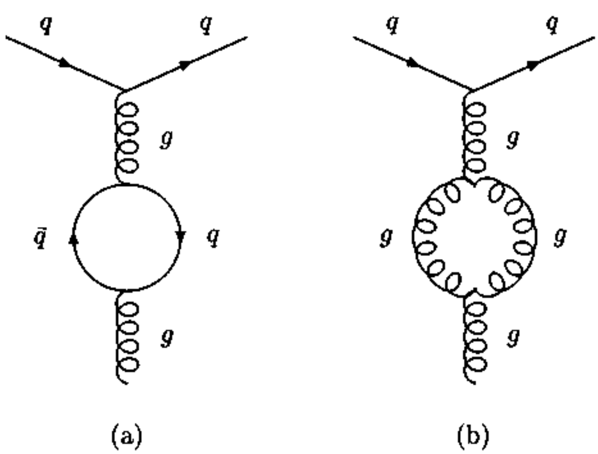


Fig. 5.12

we can easily check that to $\mathcal{O}(\alpha^2)$, (5.35) and (5.34) are equal. e_R is interptered as the measured or renormalized charge $\frac{e_R^2}{4\pi} = \alpha = \frac{1}{137}$. Here the infinity of the matrix element coming from the cut-off $\Lambda \to \infty$ is again absorbed into the observed charge e_R by renormalizing the bare charge/coupling.

5.3.2 *Running coupling constant in QCD*

As shown in (5.8), gluons have self-interactions because of the non-Abelian nature of QCD. This nature produces a drastically different behavior of the running coupling constant $\alpha_s(Q^2) = \frac{g_s^2}{4\pi}$ from the one in QED discussed above. The basic quark–gluon interaction is given in Fig. 5.11. Concerning the one–loop corrections, there are two diagrams as shown in Fig. 5.12(a) and (b). (a) shows the correction from a quark–loop which contributes equally to every quark flavor because of flavor independence. (b) shows the gluon–loop correction originated from the self-interaction of gluons. Apart from the color factor, the contribution from the quark–loop diagram (a) is essentially the same as the one in QED; it suffices to replace the factor $\frac{\alpha(\mu^2)}{3\pi}$ by $\frac{\alpha_s(\mu)}{6\pi}$ for each flavor by taking into account of the color factor. The crucial difference between QCD and QED arises from the gluon–loop diagram (b). This diagram give rise to another numerical factor, $-\frac{11}{4\pi}\alpha_s(\mu^2)$, whose sign is opposite to the one from the quark–loop. Combining the contribution from both the quark–loop (a) with n_f flavors and the gluon–loop (b), one can obtain the QCD running coupling constant $\alpha_s(Q^2)$ just by replacing the factor $\frac{\alpha(\mu^2)}{3\pi}$ in QED by $\frac{\alpha_s(\mu^2)}{4\pi}(\frac{2}{3}n_f - 11)$ as

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{(33-2n_f)\alpha_s(\mu^2)}{12\pi}\log\left(\frac{Q^2}{\mu^2}\right)}. \tag{5.36}$$

As described before, we know that there are $n_f = 6$ flavors in Nature. Putting $n_f = 6$ into (5.36), we see that the sign of the 2nd term in the denominator of (5.36) is opposite to the QED case in (5.31). Therefore, the running coupling constant $\alpha_s(Q^2)$ shows antiscreening; $\alpha_s(Q^2)$ decreases with increasing Q^2 as known as "asymptotic freedom". In contrast, for small Q^2, $\alpha_s(Q^2)$ becomes large. In fact, the demominator in (5.36) becomes 0 at some value of $Q^2 = \Lambda_{QCD}^2$ where

$$\Lambda_{QCD}^2 = \mu^2 e^{-\frac{12\pi}{(33-2n_f)\alpha_s(\mu^2)}}. \tag{5.37}$$

By using this Λ_{QCD}, we can write $\alpha_s(Q^2)$ as

$$\alpha_s(Q^2) = \frac{12\pi}{(33-2n_f)\log\frac{Q^2}{\Lambda_{QCD}^2}}. \tag{5.38}$$

Here we introduced a free parameter Λ_{QCD} with mass dimention by removing the renormalization mass parameter μ. We cannot determine the value of Λ_{QCD} theoretically in QCD. It is extracted from experiment and was determined to be $\Lambda_{QCD} \simeq 200\text{MeV}$ for $Q^2 \simeq 100\text{GeV}^2$. With $n_f = 5$ flavors (u, d, s, c, b) taking part in scattering processes, the value of α_s becomes $\alpha_s \simeq 0.2$ which justifies the perturbatuve calculation of QCD. The behavior of $\alpha_s(Q^2)$ is depicted in Fig. 5.13.

In summary, for large values of Q^2 much larger than Λ_{QCD}^2, the effective couplings between quarks and gluons becomes small and thus, the strong interaction can be treated perturbatively. In this region, quarks and gluons behave as free particles ("asymptotic freedom"). In fact, this is the region of the deep inelastic scattering scattering where the parton picture works well. On the other hand, for small Q^2 region like $Q^2 \approx \Lambda_{QCD}^2$, the quark–gluon coupling becomes large and the perturbative calculation cannot be justified. Because of the large coupling constant, all quarks and gluons are confined in hadrons ("confinement"). Λ_{QCD} is the scale which separates the world of confinement scale(hadrons) and asymptotic freedom (free quarks and gluons). QCD is the color $SU(3)$ gauge theory of strong interactions and is the fundamental field theory describing the dynamics of quarks

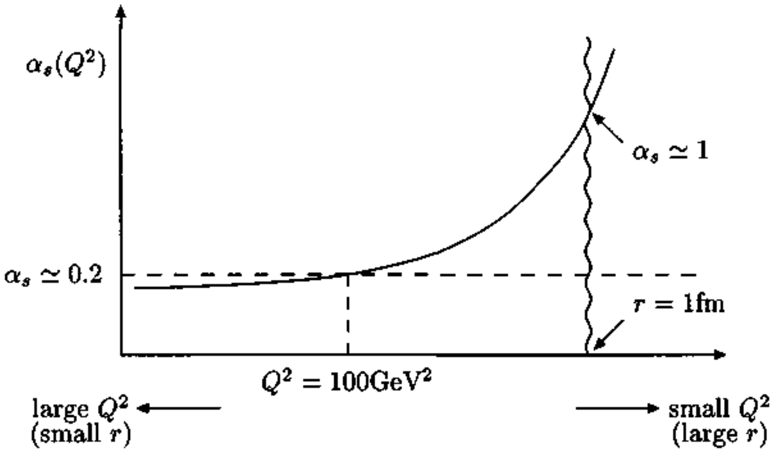


Fig. 5.13

and gluons, though the whole understandings on the nonperturbative confinement problem still incomplete.

Problems

5.1 τ^- can decay into pairs of $(e^- \ \bar{\nu}_e)$, $(\mu^- \ \bar{\nu}_\mu)$, $(d \ \bar{u})$, and $(s \ \bar{u})$; among these decay modes, the contribution of $(s \ \bar{u})$ is very small because it is a Cabibbo-suppressed process and can be neglected. Neglecting the masses of all decay particles, estimate the branching ratio, $Br = \frac{\Gamma(\tau^- \to e^- \bar{\nu}_e \nu_\tau)}{\Gamma(\tau^- \to all)}$, to be $\frac{1}{5}$ as given in (5.7).

5.2 In the tree level, the color-dependent coupling factor for the process $q \to q + g$, where q and g represent a quark and a gluon, respectively, is given by (5.18). Then, estimate the factor for $g \to q + \bar{q}$.

5.3 Consulting with other textbooks, for example, the one by Bjorken and Drell (Bjorken and Drell, 1964), derive (5.24).

Chapter 6

NEUTRINO MASSES AND NEUTRINO OSCILLATIONS

6.1 Type of Fermions and Fermion Masses

When P.A.M. Dirac first invented the spinor to describe the electron, it was a 4-component complex vector. Fermions described by such 4-component spinors are called Dirac fermions. Though the electron has a finite mass, it may be worthwhile considering a massless fermion. In this case a new feature arises: a state with $h = 1$ (h being the helicity of the particle) and a state with $h = -1$ never mix with each other during the propagation of the fermion. This can be understood intuitively by considering a inertia frame, observing the fermion co-moving toward the direction of the fermion's momentum. If the helicity h of the fermion is 1 in that frame, in any frame the helicity should be the same, just because the massless fermion is moving with the light velocity and any observer never can pass through it to invert the direction of momentum and therefore the helicity of the fermion. It is well-known that for massless fermion, the eigenstates of helicity and chiral fermions with definite eigenvalues of γ_5, i.e. Weyl fermions ψ_R and ψ_L, are identical. We easily know that each of chiral fermion forms an irreducible representation of Lorentz transformation $SL(2,C)$, as $[\gamma_5, \Sigma^{\mu\nu}] = 0$ ($\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ are 6 generators of $SU(2,C)$). In such a sense, Weyl fermions are more fundamental fermions than Dirac fermions. Let us recall that fermions to start with in the Standard Model are Weyl fermions, essentially because the weak interaction maximally breaks parity symmetry.

The statement above becomes more explicit if we use the chiral representation of γ matrices, where γ_5 is a diagonal matrix (see also Appendix A):

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{6.1}$$

$$\sigma^\mu = (-I, \sigma_i), \tag{6.2}$$

$$\bar{\sigma}^\mu = (-I, -\sigma_i), \tag{6.3}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{6.4}$$

where I is a 2×2 unit matrix and σ_i $(i = 1,2,3)$ are Pauli matrices. In the chiral representation a right- and a left-handed Weyl fermions are described by 2-conponent complex spinors η_α and $\bar{\xi}^{\dot{\alpha}}$:

$$\psi_R = \begin{pmatrix} \eta_\alpha \\ 0 \end{pmatrix}, \tag{6.5}$$

$$\psi_L = \begin{pmatrix} 0 \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix}, \quad (\alpha, \dot{\alpha} = 1,2). \tag{6.6}$$

The Lorentz generators have, as we expect, a form of block diagonal

$$\Sigma^{\mu\nu} = \frac{i}{2} \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}, \tag{6.7}$$

$$\sigma^{\mu\nu} = \sigma^\mu \bar{\sigma}^\nu, \quad \bar{\sigma}^{\mu\nu} = \bar{\sigma}^\mu \sigma^\nu, \quad (\mu \neq \nu). \tag{6.8}$$

For a massive fermion moving with a speed lower than the speed of light, the helicity may change depending on the velocity of the inertia frame, observing the fermion: the chirality is no longer preserved for massive fermions. Accordingly the fermion mass term connects Weyl fermions with different chiralities:

$$m(\overline{\psi}_R \psi_L + \overline{\psi}_L \psi_R). \tag{6.9}$$

Thus the fermion suitable to describe the propagation of the massive fermion is the combination of these two Weyl fermions, i.e. a Dirac fermion ψ_D, a full 4-component complex spinor:

$$\psi_D = \psi_R + \psi_L = \begin{pmatrix} \eta_\alpha \\ \bar{\xi}^{\dot{\alpha}} \end{pmatrix}, \tag{6.10}$$

$$\psi_R = R\psi_D, \quad \psi_L = L\psi_D, \quad (R = \frac{1+\gamma_5}{2}, \ L = \frac{1-\gamma_5}{2}), \tag{6.11}$$

The free Lagrangian for the Weyl fermions is now rewritten in terms of a Dirac spinor ψ_D:

$$\overline{\psi}_R i\partial\!\!\!/ \psi_R + \overline{\psi}_L i\partial\!\!\!/ \psi_L - m(\overline{\psi}_R \psi_L + \overline{\psi}_L \psi_R) = \bar{\psi}_D (i\partial\!\!\!/ - m)\psi_D. \tag{6.12}$$

Namely, ψ_R and ψ_L are "chiral-partners" to form a full 4-component Dirac spinor. Actually the chiral partners need not to be two independent Weyl spinors. We should note that the action of "charge conjugation" for Weyl fermions changes their chiralities; e.g. for $(\psi_R)^c = -i\gamma^2(\frac{1+\gamma_5}{2}\psi)^*$,

$$R(\psi_R)^c = 0, \ L(\psi_R)^c = (\psi_R)^c, \tag{6.13}$$

i.e. $(\psi_R)^c = (\psi^c)_L$. Or, in terms of 2-component notation of Weyl fermions,

$$\psi_R = \begin{pmatrix} \eta_\alpha \\ 0 \end{pmatrix} \to (\psi_R)^c = \begin{pmatrix} 0 \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix}, \tag{6.14}$$

where $(\eta_\alpha)^* = \overline{\eta}_{\dot{\alpha}}$, $\overline{\eta}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\overline{\eta}_{\dot{\beta}}$, $\epsilon^{12} = -\epsilon^{21} = 1$. Thus chiral partner of a Weyl spinor can be the charge conjugation of the Weyl fermion itself. Combining these chiral partners the following full 4-component spinors are possible to be formed:

$$\psi_{M1} = \psi_R + (\psi_R)^c = \begin{pmatrix} \eta_\alpha \\ \overline{\eta}^{\dot{\alpha}} \end{pmatrix}, \tag{6.15}$$

$$\psi_{M2} = \psi_L + (\psi_L)^c = \begin{pmatrix} \xi_\alpha \\ \overline{\xi}^{\dot{\alpha}} \end{pmatrix}. \tag{6.16}$$

By construction these 4-component spinors $\psi_{M1,M2}$ are self-conjugate under the charge conjugation, $(\psi_{M1,M2})^c = \psi_{M1,M2}$, and are called Majorana spinors. Though Majorana spinors are 4-component spinors, the number of independent complex degrees of freedom is 2, i.e. the same as that of a Weyl spinor. Thus as far as kinetic term is concerned, where there is no mixing between chiral partners, there is no difference between Weyl and Majorana fermions and a Dirac fermion is equivalently describable in terms of either 2 Weyl or 2 Majorana spinors:

$$\overline{\psi}_D i\partial\!\!\!/ \psi_D = \overline{\psi}_R i\partial\!\!\!/ \psi_R + \overline{\psi}_L i\partial\!\!\!/ \psi_L = \frac{1}{2}(\overline{\psi}_{M1} i\partial\!\!\!/ \psi_{M1} + \overline{\psi}_{M2} i\partial\!\!\!/ \psi_{M2}). \tag{6.17}$$

In other words, for massless fermions, the propagators are just the sum of those for independent Weyl fermions.

Once fermions get masses, the type of fermion has real physical meaning. Clearly Weyl fermion is irrelevant in this case, since the mass term connects the chiral partners. What determines the type of the full 4-component spinor, i.e. Dirac or Majorana, is the choice of chiral-partners or in other words, the type of fermion mass term. In the case of ordinary mass term for Dirac fermion (Dirac type mass term),

$$-m_D\overline{\psi}_D\psi_D = -m_D(\overline{\psi}_L\psi_R + h.c.) = m_D(\xi^\alpha\eta_\alpha + h.c.), \tag{6.18}$$

the chiral-partners are two independent Weyl fermions ξ_α and η_α. We, however, need not to take independent 2-componet spinors ξ and η as chiral partners and they can be identical. Namely if we take as chiral partners the charge conjugation of each Weyl fermion the following mass terms are also possible,

$$-\frac{1}{2}m_R(\overline{(\psi_R)^c}\psi_R + h.c.) = \frac{1}{2}m_R(\eta^\alpha\eta_\alpha + h.c.), \tag{6.19}$$

$$-\frac{1}{2}m_L(\overline{(\psi_L)^c}\psi_L + h.c.) = \frac{1}{2}m_L(\xi^\alpha\xi_\alpha + h.c.). \tag{6.20}$$

These mass terms clearly violate fermion number (lepton number for, e.g., neutrinos) and are called Majorana type mass terms, while we can assign a conserved fermion number for the case of Dirac mass term. If we adopt the Majorana type mass terms listed above, the fermions suitable to describe the free Lagrangian should be Majorana fermions ψ_{M1} and ψ_{M2}:

$$\overline{\psi}_R i\partial\!\!\!/\psi_R - \frac{1}{2}m_R(\overline{(\psi_R)^c}\psi_R + h.c.) = \frac{1}{2}\overline{\psi}_{M1}(i\partial\!\!\!/ - m_R)\psi_{M1}, \quad (6.21)$$

$$\overline{\psi}_L i\partial\!\!\!/\psi_L - \frac{1}{2}m_L(\overline{(\psi_L)^c}\psi_L + h.c.) = \frac{1}{2}\overline{\psi}_{M2}(i\partial\!\!\!/ - m_L)\psi_{M2}. \quad (6.22)$$

6.2 Neutrino Masses

6.2.1 *Possible Types of Neutrino Masses*

As we have seen in the previous section, the Majorana mass terms violate fermion number. Actually the mass terms violate all possible $U(1)$ global symmetries the Weyl fermions may have,

$$\psi_R \to e^{i\phi_R}\psi_R, \quad \psi_L \to e^{i\phi_L}\psi_L, \quad (6.23)$$

ϕ_L and ϕ_R being real transformation parameters. Thus the Majorana masses are not allowed for charged particles like electron, since if it is the case $U(1)_{em}$ symmetry is violated, i.e. the charge conservation breaks down. The Majorana mass term is, therefore, allowed only for electrically neutral particles, such as neutrinos or neutral gauge fermion in supersymmetric theory, such as photino. In this chapter we focus on the case of neutrinos.

Neutrinos of course may have ordinary Dirac masses, in addition to the possible Majorana masses. Let ν_R and ν_L be right- and left-handed neutrinos. It is interesting to note that even if we consider the most general mass term for ν_R and ν_L including the Dirac mass term,

$$\mathcal{L}_m = -\frac{1}{2}m_R\overline{(\nu_R)^c}\nu_R - \frac{1}{2}m_L\overline{(\nu_L)^c}\nu_L - m_D\overline{\nu}_R\nu_L + h.c., \quad (6.24)$$

the free Lagrangian for the neutrino still can be written in terms of two Majorana neutrinos, as we now confirm. The Dirac mass term m_D in $\mathcal{L}_m$ causes a mixing between ν_R and ν_L. We thus have to "diagonalize" the whole mass term. The mass term can be neatly written by use of matrices as

$$\mathcal{L}_m = -\frac{1}{2}\begin{pmatrix} \overline{(\nu_L)^c} & \overline{\nu_R} \end{pmatrix}\begin{pmatrix} m_L & m_D \\ m_D & m_R^* \end{pmatrix}\begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix} + h.c., \quad (6.25)$$

where the property $\overline{\nu}_R\nu_L = \overline{(\nu_L)^c}(\nu_R)^c$ has been used. Thus, the mass term, as the whole, can be written in the form of Majorana type masse term, even though there is Dirac mass term as well. This is based on the observation that ν_L and $(\nu_R)^c$ are both left-handed Weyl fermions and have no essential difference, though they have opposite lepton numbers, $L = 1$ and $L = -1$. Thus every mass term should be generally written in the form of Majorana mass term, $-\overline{(\psi_{1L})^c}\psi_{2L} + h.c.$; in the case $\psi_{1L} = \psi_{2L}$ the mass term is "genuine" Majorana mass term, while in the case of Dirac mass term $\psi_{1L} \neq \psi_{2L}$. We also find that the 2×2 mass matrix above is symmetric complex matrix, not a hermitian matrix, in general. This property is due to a relation $\overline{(\nu_{iL})^c}\nu_{jL} = \overline{(\nu_{jL})^c}\nu_{iL}$ and will be valid for more general cases with arbitrary numbers of generations. Though such symmetric complex matrices are known to be diagonalized by use of a unitary matrix in general, as we will see in the following section, here we consider the case where mass parameters m_R, m_L, m_D are all real, for brevity. The eigenvalues m_s, m_a and the corresponding eigenstates ν_s, ν_a of the mass matrix

$$M_\nu = \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}, \tag{6.26}$$

are given as

$$\begin{aligned} m_s &= \frac{1}{2}\{(m_R + m_L) + \sqrt{(m_R - m_L)^2 + 4m_D^2}\}, & (6.27) \\ m_a &= \frac{1}{2}\{-(m_R + m_L) + \sqrt{(m_R - m_L)^2 + 4m_D^2}\}, & (6.28) \\ \nu_s &= \sin\theta_\nu \nu_L + \cos\theta_\nu (\nu_R)^c, & (6.29) \\ \nu_a &= i\{\cos\theta_\nu \nu_L - \sin\theta_\nu (\nu_R)^c\}, & (6.30) \end{aligned}$$

where

$$\tan 2\theta_\nu = \frac{2m_D}{m_R - m_L}. \tag{6.31}$$

Note that the eigenvalues of M_ν obtained by an orthogonal transformation are m_s and $-m_a$, not m_a. We, however, have changed the sign, $-m_a \to m_a$, by putting the extra factor i for the eigenstate ν_a, so that the eigenvalues m_s and m_a get degenerate for the case of pure Dirac mass, $m_R = m_L = 0$, and both eigenvalues become positive for the case of "seesaw" scenario. The mass term now has been diagonalized:

$$\mathcal{L}_m = -\frac{1}{2}m_s\,\overline{(\nu_s)^c}\nu_s - \frac{1}{2}m_a\,\overline{(\nu_a)^c}\nu_a + h.c.. \tag{6.32}$$

Thus, including the kinetic term, the free Lagrangian for the neutrino is now written in terms of two Majorana neutrinos obtained by combining ν_s and ν_a with their anti-particles as chiral partners, $N_s = \nu_s + (\nu_s)^c$ and $N_a = \nu_a + (\nu_a)^c$,

$$\mathcal{L}_\nu = \frac{1}{2}\{\overline{N_s}(i\partial\!\!\!/ - m_s)N_s + \overline{N_a}(i\partial\!\!\!/ - m_a)N_a\}. \tag{6.33}$$

In the case of a scalar field, charge conjugation is equivalent to the complex conjugation of the field. So a self-conjugate Majorana fermion ($\psi^c = \psi$) corresponds to a real scalar ($\phi^* = \phi$), and the general mass matrix for two Majorana particles just corresponds to a general mass-squared matrix for two real scalars, ϕ_1 and ϕ_2,

$$\mathcal{L}_{\phi m} = -\frac{1}{2}\begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix}\begin{pmatrix} m_1^2 & m_{12}^2 \\ m_{12}^2 & m_2^2 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{6.34}$$

Actually for the case of scalar fields what matters is the mass-squared matrix. To make the correspondence clear, let us take

$$M_\nu M_\nu^\dagger = \begin{pmatrix} m_L^2 + m_D^2 & m_D(m_L + m_R) \\ m_D(m_L + m_R) & m_R^2 + m_D^2 \end{pmatrix}, \tag{6.35}$$

instead of M_ν itself. It has been well-known that when the masses of two real scalar fields are degenerate, $m_1^2 = m_2^2 \equiv m^2$, $m_{12} = 0$, they can be equivalently described by a complex scalar $\phi \equiv (1/\sqrt{2})(\phi_1 + i\phi_2)$,

$$\mathcal{L}_{\phi m} = -m^2 \ \phi^* \phi. \tag{6.36}$$

The same thing happens for fermions; for the case of pure Dirac mass, $m_L = m_R = 0$, $M_\nu M_\nu^\dagger$ has degenerate eigenvalues m_D^2 and two Majorana neutrinos are equivalently described in terms of Dirac particle,

$$\nu_D = \frac{N_s - iN_a}{\sqrt{2}} = \nu_R + \nu_L \ \ (\theta_\nu = \frac{\pi}{4}), \tag{6.37}$$

whose free Lagrangian is written as

$$\mathcal{L}_\nu = \overline{\nu_D}(i\not\partial - m_D)\nu_D. \tag{6.38}$$

The factor 1/2 appearing in the free Lagrangian for Majorana neutrinos (6.32) just corresponds to the same normalization factor in the free Lagrangian for real scalars. Hence, just as in the case of scalar fields, such normalized Majorana fields N_s and N_a should have the same propagator as the one for a Dirac field. For instance, for the Majorana fermion ψ_{M1} in (6.21) the propagator without chirality flip, and therefore without lepton number violation, is given (in momentum space) as usual,

$$R \cdot \langle 0|\psi_{M1}\overline{\psi_{M1}}|0\rangle \cdot L = \langle 0|\psi_R\overline{\psi_R}|0\rangle = R \cdot \frac{i}{\not p - m_R} \cdot L. \tag{6.39}$$

This suggests that both ψ_{M1} and ψ_{M2} are properly normalized fields. Thus, generally the propagator for a Majorana field ψ_M with a mass m is given as

$$\langle 0|\psi_M\overline{\psi_M}|0\rangle = \frac{i}{\not p - m} \tag{6.40}$$

$$\langle 0|\psi_M\psi_M^t|0\rangle = \frac{i}{\not p - m} \cdot C^{-1}. \tag{6.41}$$

A characteristic feature of Majorana field is that the propagator, apparently violating the fermion number, $\langle 0|\psi_M\psi_M^t|0\rangle$, exists in addition to the ordinary one, $\langle 0|\psi_M\overline{\psi_M}|0\rangle$. In (6.41), t stands for the transpose and $C = i\gamma^0\gamma^2$ is the charge conjugation matrix. The propagator of the type $\langle 0|\psi_M\psi_M^t|0\rangle$ is needed in the calculation of e.g. neutrino-less double β-decay.

6.2.2 *The Mechanism of Neutrino Mass Generation*

Recent observations of neutrino oscillations strongly suggest that neutrinos have finite masses. These data on neutrino oscillations, however, do not tell us the type of neutrino masses, i.e. Dirac or Majorana. Though Majorana masses violate lepton number, the smallness of neutrino masses prevent us to confirm or exclude the Majorana property by use of the data of the processes sensitive to lepton number violation, whose typical example is the neutrino-less double β-decay. Thus, in the consideration of the scenario of neutrino mass generation, we do not have a priori any idea of relative strengths of Dirac and Majorana masses. Therefore, let us now consider three typical (extreme) cases, depending on the relative strengths of these masses. We will find that the smallness of neutrino masses can be naturally accounted for in the third possibility, so-called seesaw scenario.

(A) Pure Dirac

We first consider the case that neutrinos have only Dirac masses, $m_R = m_L = 0$, and neutrinos are Dirac particles. As we have already seen above, in this case a Dirac neutrino may be regarded as the superposition of two Majorana neutrinos N_s and N_a with degenerate masses, $m_s = m_a = m_D$ (see (6.27), (6.28)). The mixing angle is "maximal", i.e. $\theta_\nu = \pi/4$, as can be seen in (6.31). This maximal mixing, however, does not lead to any neutrino oscillation, though it mimics the flavor or generation mixing; as we will discuss, neutrino oscillation necessitates not only a mixing angle but also a mass(-squared) difference. In this pure Dirac case, we thus need flavor mixings among different generations to realize the neutrino oscillations, as will be discussed in the next section. It is interesting to note that though each of the two Majorana neutrinos N_s, N_a does contribute to the lepton-number violating neutrino-less double β-decay, their contributions, obtained by use of the propagator (6.41), exactly cancel with each other, due to the factor i in (6.30). This should be the case, since in pure Dirac case the lepton number should be preserved. In this case, however, we have no special reason to expect that the Dirac masses of neutrinos should be much smaller than the Dirac masses of corresponding charged particles, such as electron. So a serious problem of why neutrinos masses are so small remains to be solved.

(B) Pseudo-Dirac

We next consider what happens if we put small Majorana masses, m_R, $m_L \ll m_D$. The neutrinos are still almost Dirac particle and are called pseudo-Dirac neutrinos (Wolfenstein, 1981). The mixing angle is still almost maximal, i.e. $\theta_\nu \simeq \pi/4$. One remarkable feature is that in this case there appears a slight difference of mass eigenvalues $m_s^2 - m_a^2 \simeq 2m_D(m_R + m_L)$. Together with the almost maximal mixing, this mass-squared difference leads to a neutrino oscillation even if there is only 1 generation (Kobayashi and Lim, 2001). In fact a neutrino oscillation $\nu_L \to (\nu_R)^c$ occurs with almost maximal mixing. The probability for the neutrino produced as

ν_L at time 0 with energy E to be observed as $(\nu_R)^c$ at time t, in the simplified 1 generation scheme, is known to be given as

$$P(\nu_L \to (\nu_R)^c) \simeq \sin^2(\frac{m_D(m_R + m_L)t}{2E}). \tag{6.42}$$

One interesting feature of the neutrino oscillations of pseudo-Dirac neutrinos is that the maximal mixing, strongly suggested by the Super-Kamiokande data on the atmospheric neutrino oscillation, is almost automatically realized. Such neutrino oscillations, however, are those into "sterile" states $(\nu_R)^c$ without electro-weak interaction, which are not favored by the data of Super-Kamiokande experiments. Again, the problem of smallness of neutrino masses cannot be naturally solved in this scenario.

(C) Seesaw (Yanagida, 1979; Gell-Mann–Ramond–Slansky, 1979)
We finally consider another extreme case where Majorana masses are much larger than the Dirac mass. We, however, find that the Majorana mass for left-handed neutrino, m_L, cannot be large, while the Majorana mass for right-handed neutrino, m_R, can be sufficiently large. This difference essentially comes from the difference of the transformation property of each Majorana mass term under $SU(2)_L \times U(1)_Y$, the gauge group of the standard model. We immediately know that the right-handed Majorana mass term $m_R\overline{(\nu_R)^c}\nu_R$ is gauge invariant under $SU(2)_L \times U(1)$, as ν_R is a $SU(2)_L$ singlet neutral particle. On the other hand, the left-handed Majorana mass term $m_L\overline{(\nu_L)^c}\nu_L$, is not gauge invariant, as ν_L belongs to a $SU(2)_L$ doublet, and this bare mass term is not allowed in the Lagrangian. In fact, we know that the product of the fields $\overline{(\nu_L)^c}\nu_L$ should belong to either singlet or triplet of $SU(2)_L$ (doublet $\times$ doublet $=$ singlet $+$ triplet). Actually it cannot be the singlet, since $\overline{(\nu_L)^c}\nu_L$ behaves as $I_3 = 1$. Thus $\overline{(\nu_L)^c}\nu_L$ should behave as a component of $SU(2)_L$ triplet, and to form a gauge invariant (renormalizable) Yukawa interaction, we need a Higgs H_T belonging to $SU(2)_L$ triplet, so that $m_L \sim f\langle H_T\rangle$ with f being the Yukawa coupling. The VEV $\langle H_T\rangle$, however, cannot be large. If the VEV is larger than or comparable to the VEV of ordinary $SU(2)_L$ doublet Higgs, the famous relation, well-established experimentally,

$$\rho \equiv \frac{M_W^2}{M_Z^2\cos^2\theta_W} \simeq 1, \tag{6.43}$$

no longer holds at the classical (tree) level. On the other hand there is no good reason to expect that m_R, being $SU(2)_L$ invariant, should be small. Rather, it will be natural to expect that it is much larger than the VEV of the doublet Higgs, i.e. than the weak scale M_W, since it is quite possible that the m_R actually comes from the energy scale of some "new physics" beyond the standard model, which is expected much higher than M_W. The above argument suggests that $m_L \ll m_D \ll m_R$. Here, for brevity, let us assume that H_T is absent and therefore

$m_L = O$ $(m_D \ll m_R)$. In this case the neutrino mass matrix reduces to

$$M_\nu = \begin{pmatrix} 0 & m_D \\ m_D & m_R \end{pmatrix}. \tag{6.44}$$

One immediately knows that one mass eigenvalue of M_ν, $m_s \simeq m_R$ under $m_D \ll m_R$, and another eigenvalue

$$m_a \simeq \frac{m_D^2}{m_R} \ll m_D, \tag{6.45}$$

since the product of two eigenvalues is given by $\det M_\nu = -m_D^2$. (Strictly speaking this means $m_a \simeq -\frac{m_D^2}{m_R}$, with opposite sign, but the extra minus sign can be absorbed by putting a phase i to the eigenstate ν_a, as is seen in (6.30).) These eigenvalues may be obtained from (6.27) and (6.28), as well. We thus have obtained a naturally small ($\ll m_D$) neutrino mass m_a by making m_R much larger than m_D. The relation $m_s \cdot m_a \simeq m_D^2$ suggests the terminology seesaw. Namely, when one eigenstate N_s gets heavy another eigenstate N_a is "lifted". We know that under $m_D \ll m_R$ the mixing angle θ_ν is small

$$\theta_\nu \simeq \frac{m_D}{m_R} \ll 1 \tag{6.46}$$

$$N_s \simeq \nu_R + (\nu_R)^c, \quad N_a \simeq i\{\nu_L - (\nu_L)^c\}. \tag{6.47}$$

As the heavier Majorana neutrino N_s decouples from lower energy processes, only $N_a \simeq i\{\nu_L - (\nu_L)^c\}$ will remain and there will be no neutrino oscillation in one generation scheme. Thus, as in the case of pure Dirac scenario, to have neutrino oscillation, flavor or generation mixing is inevitable.

One may wonder, in the absence of the triplet Higgs H_T, which operator can provide the small Majorana mass m_a, which is effectively the Majorana mass for the left-handed neutrino, as $N_a \simeq i\{\nu_L - (\nu_L)^c\}$. It turns out that though there is no renormalizable operator in the original Lagrangian relevant for the Majorana mass, a higher dimensional ($d = 5$) "irrelevanth effective operator,

$$\frac{c}{M}\,\phi^t \epsilon\sigma_a \phi \cdot \begin{pmatrix} \nu_L^t & e_L^t \end{pmatrix} \epsilon\sigma_a\, C \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \tag{6.48}$$

plays the role, where $\phi = (\varphi^+, \varphi^0)^t$ is the Higgs doublet and ϵ is 2×2 anti-symmetric matrix ($\epsilon^{12} = -\epsilon^{21} = 1$) and C is the charge conjugation matrix. The coefficient c is a dimensionless coupling and if we regard $c = f_D^2$ (f_D: Yukawa coupling in the Dirac mass term), $M = m_R$ and replace φ^0 by the VEV $v/\sqrt{2}$, we get a left-handed Majorana mass, $\sim \frac{f_D^2 v^2}{m_R} \sim \frac{m_D^2}{m_R}$, in accordance with the result obtained by the diagonalization of the mass matrix. In fact, the diagram shown in Fig. 6.1 is known to provide (a part of) the effective higher-dimensional operator, when the momentum is ignored compared with m_R in the propagator of the internal line.

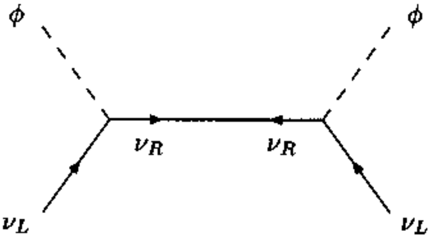


Fig. 6.1

When M goes to infinity the effective operator vanishes. Thus the essence of seesaw may be understood as the decoupling phenomenon of a heavy particle with mass M, which is right-handed Majorana neutrino N_s in the case of seesaw mechanism. The importance of the argument in terms of the effective operator is that the M needs not to come from the right-handed Majorana neutrino, but may be attributed to any heavy $SU(2)$ singlet field with mass M.

6.3 Flavor Mixing and Neutrino Oscillation (in the Vacuum)

If we include right-handed neutrinos to accommodate neutrino masses, the lepton sector is described by 3 doublets and 6 singlets of $SU(2)_L$:

$$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}; \quad \nu_{eR}, \ \nu_{\mu R}, \nu_{\tau R}, e_R, \mu_R, \tau_R. \tag{6.49}$$

Without loss of generality, the charged leptons, e_L, e_R etc., may be understood as their mass eigenstates, and their partners in the doublets, ν_{eL} etc. are states emitted by weak interaction processes and are called "weak eigenstates". Suppose the mass matrix for neutrinos in the base of weak eigenstates is diagonal. (For brevity we here assume neutrinos have only Dirac masses), Then, the generation number or quantum number of flavor such as electron number L_e ($L_e = 1$ for e^-, ν_e, $L_e = 0$ for others), muon-number L_μ etc. are strictly conserved. If, on the other hand, the mass matrix is non-diagonal, the flavor changing processes such as $\nu_e \to \nu_\mu$ or $\mu \to e\gamma$ will become possible. The phenomenon such as $\nu_e \to \nu_\mu$, where a neutrino of some specific flavor is transformed into a neutrino of another flavor, is called "neutrino oscillation", since the probability of the transition oscillates as the function of time. Even if we switch on flavor mixing, provided the neutrino masses are all vanishing or, more generally, degenerate, the amplitudes of flavor changing processes will exactly vanish. This can be explicitly checked in the formula of neutrino oscillation probabilities, as we will see later (see for instance (6.69) for $\Delta m^2_{21} = 0$). This fact also may be easily understood from a symmetry argument. When neutrino masses are all degenerate, there appears a global symmetry $U(3)$ among 3 generations or flavors in neutrino sector, and by use of Noether's theorem, we easily find that there should be 3 conserved additive quantum numbers, which may be interpreted as L_e etc. Or, more intuitively, if neutrino masses are all degenerate there would be nothing to distinguish neutrino flavors, and we may always perform a unitary transformation, so that all flavor mixing angles vanish in the new base. Thus to realize flavor changing processes we need both flavor (or generation) mixing and non-degenerate neutrino masses. This is in complete similarity with the content of GIM mechanism in quark sector.

It should be emphasized that now there are increasing evidences for neutrino oscillations claimed by the experiments to detect the solar and atmospheric neutrinos, as will be discussed later in this chapter. The phenomenon of neutrino oscillation is quite interesting by its own right. In addition, theoretically, it is expected to play a central role in the search for and the establishment of "new physics", since it clearly indicates non-vanishing neutrino masses and therefore some physics beyond the standard model.

A nice comprehensive review on the neutrino masses and neutrino oscillations is given, e.g., in the recent textbook by Fukugita and Yanagida (Fukugita-Yanagida, 2003).

6.3.1 *Flavor Mixing*

We now discuss the flavor mixing, the origin of neutrino oscillation, for two typical scenarios, (A) pure Dirac, and (B) seesaw.

(A) Flavor mixing in the pure Dirac scenario

Just as in quark sector, if neutrinos have only Dirac masses, the mass term for weak eigenstates $(\nu_{eL},\ \nu_{\mu L},\ \nu_{\tau L})$ and $(\nu_{eR},\ \nu_{\mu R},\ \nu_{\tau R})$ is given as

$$\mathcal{L}_m = (m_D)_{\alpha\beta}\overline{\nu_{\alpha L}}\nu_{\beta R} + h.c. \quad (\alpha,\ \beta = e,\ \mu,\ \tau). \tag{6.50}$$

Though a Dirac fermion can be decomposed into two Majorana fermions with degenerate masses as we have already seen in the previous section, we do not take this picture, and the mass matrix is 3×3 matrix just as in the quark sector. As we have learned in the quark sector (see (4.144), (4.145)), the matrix m_D can be diagonalized by a bi-unitary transformation,

$$U^\dagger m_D V = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \tag{6.51}$$

where U and V are unitary matrices. The left-handed eigenstates of mass matrix, i.e. mass-eigenstates, $\nu_{1L},\ \nu_{2L},\ \nu_{3L}$ with Dirac masses, $m_1,\ m_2,\ m_3$ are related to weak eigenstates as

$$\begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = U \cdot \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}. \tag{6.52}$$

Accordingly the charged current interaction of left-handed leptons is written in terms of U

$$\mathcal{L}_c = \frac{g}{\sqrt{2}} \begin{pmatrix} \overline{e_L} & \overline{\mu_L} & \overline{\tau_L} \end{pmatrix} U \gamma_\mu \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix} \cdot W^{-\mu}, \tag{6.53}$$

and the unitary matrix, corresponding to the Kobayashi-Maskawa matrix in quark sector, is called Maki–Nakagawa–Sakata (MNS) matrix (Maki, Nakagawa and Sakata, 1962). In the $SU(2)_L \times U(1)_Y$ gauge theory, only left-handed neutrinos have weak interactions, and as we will see in the following section, only neutrino oscillations without chirality-flip are important. These lead to the conclusion that not m_D itself, but only the combination $m_D m_D^\dagger$ is relevant for the physical processes, including neutrino oscillation. Since

$$m_D m_D^\dagger = U \text{ diag}(m_1^2, m_2^2, m_3^2)\ U^\dagger, \tag{6.54}$$

the unitary transformation of right-handed neutrinos V never appears in the physical observables.

(B) Flavor mixing in the seesaw scenario

Now with 3 generations the seesaw mechanism should be implemented in the 6×6 mass matrix. In the base of $\psi = (\nu_{eL},\ \nu_{\mu L},\ \nu_{\tau L}, (\nu_{eR})^c,\ (\nu_{\mu R})^c,\ (\nu_{\tau R})^c)^t$, the 6×6 mass matrix m_ν takes the form of

$$m_\nu = \begin{pmatrix} m_L & m_D^t \\ m_D & m_R^* \end{pmatrix}, \tag{6.55}$$

where 3×3 matrices m_L, m_R, m_D denote left-handed, right-handed Majorana mass matrices and Dirac mass matrix, respectively. As we have already seen in the simplified 1 generation scheme, the mass matrix m_ν, as the whole, is a symmetric matrix, $m_\nu^t = m_\nu$ $(m_L^t = m_L, m_R^t = m_R)$. To consider the seesaw scenario, we assume that $m_L = 0$. The mass matrix with $m_L = 0$ is easily achieved in the standard model, by introducing right-handed neutrinos and their Yukawa couplings and bare mass terms,

$$-f_{\alpha\beta}^D \begin{pmatrix} \overline{\nu_{\alpha L}} & \overline{l_{\alpha L}^-} \end{pmatrix} \tilde{\phi} \nu_{\beta R} - (m_R)_{\alpha\beta} \overline{(\nu_{\alpha R})^c} \nu_{\beta R}, \tag{6.56}$$

where $(m_D)_{\alpha\beta} = f_{\beta\alpha}^{D*} \cdot \frac{v}{\sqrt{2}}$, $\tilde{\phi} \equiv i\tau_2 \phi^*$ (see (4.137)) and $\frac{v}{\sqrt{2}}$ is the VEV of the neutral Higgs φ^0. To account for the seesaw mechanism we understand that all matrix elements of m_R are much larger than those of m_D. Then, as we have seen in the 1 generation scheme, the mutual mixings among $\nu_{\alpha L}$ and $(\nu_{\alpha R})^c$ (corresponding to θ_ν in (6.29), (6.30)) are small, being suppressed by the ratios of elements of m_D to those of m_R. Just as in the 1 generation scheme, the mass matrix is made into the form of block-diagonal by a suitable unitary transformation with the small mixing angles,

$$\begin{aligned} &\begin{pmatrix} iI & -im_D^t m_R^{*-1} \\ m_R^{-1} m_D^* & I \end{pmatrix} m_\nu \begin{pmatrix} iI & m_D^\dagger m_R^{-1} \\ -im_R^{*-1} m_D & I \end{pmatrix} \\ \simeq\ &\begin{pmatrix} m_D^t (m_R^*)^{-1} m_D & 0 \\ 0 & m_R^* \end{pmatrix}. \end{aligned} \tag{6.57}$$

Thus again eigenstates with masses of the order of m_R, decouple from the low energy effective theory, and the mass matrix for the remaining neutrinos, which are approximately $\nu_{\alpha L}$, may be regarded as

$$m_{\nu L} = m_D^t (m_R^*)^{-1} m_D, \tag{6.58}$$

which is again a symmetric matrix.

It is mathematically proven that a symmetric complex matrix can be diagonalized with real eigenvalues by a unitary transformation,

$$U^t m_{\nu L} U = \mathrm{diag}(m_1,\ m_2,\ m_3). \tag{6.59}$$

Again what matters in the neutrino oscillation is not $m_{\nu L}$ itself, but

$$m_{\nu L}^\dagger m_{\nu L} = U\ \mathrm{diag}(m_1^2, m_2^2, m_3^2)\ U^\dagger. \tag{6.60}$$

It is quite important to note that the right-hand-side of the equation is exactly the same as the corresponding one in pure Dirac scenario, (6.54). Thus as far as our concern is neutrino oscillation, to discriminate the scenarios of neutrino mass generation, pure Dirac or seesaw, is impossible. Note, however, if we observe a process with chirality-flip, $\nu_L \to \nu_R$ or $\nu_L \to (\nu_L)^c$, it is in principle possible to discriminate these two scenarios, since the final states ν_R and $(\nu_L)^c$ have opposite lepton numbers and only $(\nu_L)^c$ has weak interaction. The typical and interesting example with chirality flip and lepton number violation is the neutrino-less double β-decay. If the process is observed, it will clearly indicate the Majorana property suggested by the seesaw scenario, though the decay amplitude is suppressed by the small Majorana neutrino masses of ν_L, and the detection is not easy.

6.3.2 *Neutrino Oscillation (in the Vacuum)*

Now let us move to the discussion on the neutrino oscillation in the vacuum. (Neutrino oscillation in the presence of matter will be discussed in the next section.) The neutrino oscillation is known for long time to be the most natural framework to explain the lower than expected capture rates of solar and atmospheric neutrinos, i.e. the puzzles of solar and atmospheric neutrinos, which we will discuss later in this chapter. We have argued that neutrino oscillation is possible, for pure Dirac or seesaw scenarios, only if there are more than 1 generation of leptons. Strictly speaking even in 1 generation scheme, there exists an "oscillation" of the type "$\nu_L \to \nu_R$" with chirality flip. This seems to be able to explain, just with 1 generation, the puzzle of e.g. solar neutrino, since the transformed state ν_R has no weak interaction (a "sterile" state) and escapes the detection. However, unfortunately this is not the case. As the neutrino masses are known to be quite small, neutrinos are highly relativistic under the real situation of experimental detection, i.e. $m_\nu \ll E_\nu$. Since for the relativistic particle the chirality is nearly equal to the helicity and therefore is almost preserved, the probability of the transition $\nu_L \to \nu_R$ is quite small,

$$P(\nu_L \to \nu_R) \sim (\frac{m_\nu}{E_\nu})^2 \ll 1. \tag{6.61}$$

The fact that the chirality flip is negligible suggests that in the consideration of neutrino oscillation the spin degree of freedom is not relevant. (If we take the effect of background magnetic field into account, which may be suitable for neutrinos propagating inside the sun or supernovae, the spin-precession of the neutrinos due to the anomalous magnetic moment may take place without the chiral suppression factor $(m_\nu/E_\nu)^2$ (Voloshin et al., 1988; Lim and Marciano, 1988; Akhmedov, 1988), and this statement does not hold. We, however, ignore here the effect of the magnetic field.) Thus the relevant equation of motion for neutrinos is the Klein-Gordon equation, rather than the Dirac equation. We have already discussed that when the chirality flip can be ignored there is no essential difference, concerning neutrino oscillations, in the two cases of pure Dirac and seesaw. Thus we now assume that neutrinos are pure Dirac particles. The mass eigenstates ν_{iL} ($i = 1, 2, 3$) obey the following Klein-Gordon equation

$$(\Box + m_i^2)\ \nu_i = 0, \tag{6.62}$$

whose solution with definite momentum and (positive) energy is just an ordinary plane-wave solution

$$\nu_i = e^{-ip_{i\mu}\cdot x^\mu} = e^{-iE_i t} \cdot e^{i\vec{p}\cdot\vec{x}}, \tag{6.63}$$

where $E_i = \sqrt{\vec{p}^2 + m_i^2}$ and the 3-momentum $\vec{p}$ is taken to be common for all ν_i for convenience. For relativistic neutrinos, $|\vec{p}| \gg m_i$, the approximation $E_i \simeq |\vec{p}| + (m_i^2/2|\vec{p}|) \simeq |\vec{p}| + (m_i^2/2E)$ (E: the average neutrino energy) is valid. Furthermore, the factor $e^{-i|\vec{p}|t} \cdot e^{i\vec{p}\cdot\vec{x}}$ is an overall phase factor, common for all neutrinos, and does not affects the physical observables. Thus if we regard the matter waves of neutrinos as $\nu_i(t) = e^{-i(m_i^2/2E)}$, they obviously satisfy the following equation of motion

$$i\frac{d}{dt}\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} \frac{m_1^2}{2E} & 0 & 0 \\ 0 & \frac{m_2^2}{2E} & 0 \\ 0 & 0 & \frac{m_3^2}{2E} \end{pmatrix} \cdot \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \tag{6.64}$$

As neutrinos are emitted and observed as weak-eigenstates, not as mass-eigenstates, it will be useful to consider the time-evolution equation in the base of weak-eigenstates. The (6.52), (6.54) tells us that the differential equation is obtained by a unitary transformation with the MNS matrix U:

$$\begin{aligned} i\frac{d}{dt}\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} &= U\cdot\begin{pmatrix} \frac{m_1^2}{2E} & 0 & 0 \\ 0 & \frac{m_2^2}{2E} & 0 \\ 0 & 0 & \frac{m_3^2}{2E} \end{pmatrix}\cdot U^\dagger\cdot\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \\ &= \frac{1}{2E}\cdot m_\nu m_\nu^\dagger\cdot\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}. \end{aligned} \tag{6.65}$$

As we have seen above this expression holds for both of pure Dirac and seesaw scenarios. This is why we cannot discriminate these two types of neutrino mass generation just from the data of neutrino oscillations. The above evolution equation is readily solved to be

$$\begin{aligned} \begin{pmatrix} \nu_e(t) \\ \nu_\mu(t) \\ \nu_\tau(t) \end{pmatrix} &= \exp(-\frac{i}{2E}m_\nu m_\nu^\dagger t)\cdot\begin{pmatrix} \nu_e(0) \\ \nu_\mu(0) \\ \nu_\tau(0) \end{pmatrix} @ \\ &= U\begin{pmatrix} e^{-i\frac{m_1^2}{2E}t} & 0 & 0 \\ 0 & e^{-i\frac{m_2^2}{2E}t} & 0 \\ 0 & 0 & e^{-i\frac{m_3^2}{2E}t} \end{pmatrix}U^\dagger\begin{pmatrix} \nu_e(0) \\ \nu_\mu(0) \\ \nu_\tau(0) \end{pmatrix}, \end{aligned} \tag{6.66}$$

where a relation $exp(-\frac{i}{2E}m_\nu m_\nu^\dagger t) = exp(-\frac{i}{2E}Um^2_{\text{diag}}U^\dagger t)$
$= Uexp(-\frac{i}{2E}m^2_{\text{diag}}t)U^\dagger$ ($m^2_{\text{diag}} = \text{diag}(m_1^2, m_2^2, m_3^2)$) has been used.

Suppose a neutrino is born as ν_α at time 0. The initial condition is then given as $\nu_\alpha(0) = 1$, others $= 0$. Under this initial condition, the probability amplitude to detect ν_β at time t becomes $\nu_\beta(t) = \sum_i U_{\beta i}\, e^{-i\frac{m_i^2}{2E}t}\, U^*_{\alpha i}$. Thus the probability for the neutrino born as ν_α at time 0 to be observed as ν_β at time t is given by a formula (see Fig. 6.2)

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) &= |\sum_i U_{\beta i}\, e^{-i\frac{m_i^2}{2E}t}\, U^*_{\alpha i}|^2 \\ &= |\sum_i U_{\beta i}\, e^{-i\frac{\Delta m_{i1}^2}{2E}t}\, U^*_{\alpha i}|^2. \end{aligned} \tag{6.67}$$

In the second line of (6.67) $\Delta m_{i1}^2 = m_i^2 - m_1^2$ ($\Delta m_{11}^2 = 0$) and an overall phase factor has been ignored. This formula clearly shows the fact that to get neutrino oscillation both flavor mixing described by U and the mass(-squared) differences Δm^2 are necessary. In fact, if $\Delta m_{21}^2 = \Delta m_{31}^2 = 0$ then $P(\nu_\alpha \to \nu_\beta) = 0$ for $\alpha \neq \beta$ due to the unitarity $(UU^\dagger)_{\beta\alpha} = 0$. More intuitively, the neutrino oscillation

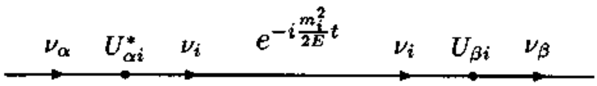


Fig. 6.2

is purely quantum mechanical effect and may be understood as a sort of "beat" of (coherent) matter waves describing mass eigenstates ν_i. The beat, of course, needs slight differences of frequencies, i.e. slight mass differences of ν_i (for a fixed momentum $\vec{p}$ to realize the coherence). Though we have supposed that there are 3 generations, actually the above formula is valid for arbitrary number of generations.

One remark here is concerning CP violating phases. In the case of pure Dirac neutrinos the physically meaningful degrees of freedom of CP violating phases in the MNS matrix U is $\frac{(n-1)(n-2)}{2}$ (n : the number of generations), just as in the argument of Kobayashi-Maskawa matrix in the quark sector. In the case of Majorana neutrinos suggested by the seesaw scenario, however, it is known that the argument needs some modification. Namely, the re-phasing of (left-handed) neutrino states in order to eliminate some phases in U defined in (6.59) is no longer possible, while the re-phasing of charged leptons goes just as in the quark sector. This is because the Majorana mass term for mass-eigenstates ν_i, $-\frac{1}{2}m_i\ \overline{(\nu_{iL})^c}\ \nu_{iL}$, is not invariant under the re-phasing $\nu_{iL} \to e^{i\phi_i}\ \nu_{iL}$, as the mass term violates U(1) symmetries, especially the lepton number, in general. Thus the number of physically meaningful CP-violating phases is modified into $n^2 - \frac{n(n-1)}{2} - n = \frac{n(n-1)}{2}$. The difference $\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n-1$ denotes the number of newly added CP phases for Majorana neutrinos and these phases are called Majorana-phases. Thus even in 2 generation we have CP violation, in principle. Such Majorana-phases, however, does not appear in neutrino oscillation processes. This is simply because the difference between pure Dirac and seesaw (or Majorana) scenarios does not manifest itself in the neutrino oscillations. The essence of this statement is that even if there may appear a phase $e^{2i\phi_i}m_i$ by the re-phasing of neutrinos, what matters in the oscillation probability is the combined factor $(e^{2i\phi_i}m_i)(e^{2i\phi_i}m_i)^* = m_i^2$, not the mass itself. Recall that the neutrino oscillation we are considering is the one without chirality-flip. This suggests that the Majorana phases are really meaningful in the processes with chirality flip and the violation of lepton number, whose typical example is neutrino-less double β-decay. In fact, the factor i appearing in (6.30) is a sort of Majorana phase, and we have seen there that the phase played a role to cancel the amplitude of the neutrino-less double β-decay in the pure Dirac scenario.

Though there exist 3 generations, in reality, for illustrative purpose let us discuss for a while the simplified 2 generation scheme, assuming that $\nu_\alpha = \nu_e,\ \nu_\mu$ and $\nu_i = \nu_1,\ \nu_2$. We may assume without loss of generality that the mixing matrix U is real orthogonal matrix, as there does not appear any CP violating phase for $n = 2$:

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{6.68}$$

In this case, the mass-squared difference Δm^2_{21} is unique. Then (6.67) tells us that the transition probability of the process $\nu_e \to \nu_\mu$ reads as

$$P(\nu_e \to \nu_\mu) = \sin^2 2\theta \, \sin^2(\frac{\Delta m^2_{21}}{4E} t). \tag{6.69}$$

As we expected, $P(\nu_e \to \nu_\mu) = 0$ for $\theta = 0$ or $\Delta m^2_{21} = 0$. On the other hand the "survival probability" of ν_e is given as

$$\begin{aligned} P(\nu_e \to \nu_e) &= \cos^4\theta + \sin^4\theta + 2\cos^2\theta\sin^2\theta\cos(\frac{\Delta m^2_{21}}{2E} t) \\ &= 1 - \sin^2 2\theta \, \sin^2(\frac{\Delta m^2_{21}}{4E} t). \end{aligned} \tag{6.70}$$

These probabilities satisfy a relation

$$P(\nu_e \to \nu_e) + P(\nu_e \to \nu_\mu) = 1, \tag{6.71}$$

which reflects the conservation of probability.

For an arbitrary number of generations, the CPT theorem implies

$$P(\nu_\alpha \to \nu_\beta) = P(\overline{\nu_\beta} \to \overline{\nu_\alpha}). \tag{6.72}$$

If CP is an exact symmetry, we further get

$$P(\nu_\alpha \to \nu_\beta) = P(\overline{\nu_\alpha} \to \overline{\nu_\beta}). \tag{6.73}$$

If T is an exact symmetry, we get

$$P(\nu_\alpha \to \nu_\beta) = P(\nu_\beta \to \nu_\alpha). \tag{6.74}$$

(6.73) is obtainable from (6.72) and (6.74) as the CP-invariance is equivalent to T-invariance, under the CPT theorem. In the two generation case there is no CP violating effect, and all of these relations hold. (These relations are spoilt, in general, once the matter effects, discussed in the next section, are taken into account.) For instance,

$$P(\nu_\mu \to \nu_e) = P(\nu_e \to \nu_\mu), \tag{6.75}$$

and therefore the unitarity for ν_e, (6.71), and the corresponding relation for ν_μ imply

$$P(\nu_\mu \to \nu_\mu) = P(\nu_e \to \nu_e). \tag{6.76}$$

In realistic experimental situation, it is quite possible that the wave length of the neutrino oscillation is much smaller than the uncertainties of the positions of production and detection points of the neutrinos. If it is the case, the relevant quantities are time-averaged (the average over the positions is equivalent to the time-average) transition or survival probabilities. For instance, in the 2 generation scheme, we have the time-averaged probabilities denoted by $\bar{P}$,

$$\overline{P}(\nu_e \to \nu_\mu) = \frac{1}{2}\sin^2 2\theta, \tag{6.77}$$

$$\overline{P}(\nu_e \to \nu_e) = 1 - \frac{1}{2}\sin^2 2\theta. \tag{6.78}$$

Though the survival probability (6.70) itself can be sufficiently small for $\theta \simeq \pi/4$, the time-averaged probability has a lower bound,

$$\overline{P}(\nu_e \to \nu_e) \geq \frac{1}{2}. \tag{6.79}$$

This can be easily generalized to the scheme of arbitrary number of generations n:

$$\overline{P}(\nu_e \to \nu_e) \geq \frac{1}{n}. \tag{6.80}$$

To prove this inequality, we first notice that when time-average is taken, 6.67 provides us

$$\overline{P}(\nu_\alpha \to \nu_\beta) = \sum_i |U_{\beta i}|^2 |U_{\alpha i}|^2. \tag{6.81}$$

In particular, the averaged survival probability of ν_α is simply given as

$$\overline{P}(\nu_\alpha \to \nu_\alpha) = \sum_i |U_{\alpha i}|^4. \tag{6.82}$$

Then the inequality is easily proven by use of the unitarity condition of the MNS matrix, $\sum_i |U_{\alpha i}|^2 = 1$. The square of this unitariry relation yields

$$\sum_i |U_{\alpha i}|^4 + 2\sum_{i<j} |U_{\alpha i}|^2 |U_{\alpha j}|^2 = 1. \tag{6.83}$$

On the other hand, we have a trivial relation

$$\sum_{ij} (|U_{\alpha i}|^2 - |U_{\alpha j}|^2)^2 \geq 0, \tag{6.84}$$

which gives

$$(n-1)(\sum_i |U_{\alpha i}|^4) - 2\sum_{i<j} |U_{\alpha i}|^2 |U_{\alpha j}|^2 \geq 0. \tag{6.85}$$

Summing up (6.83) and (6.85), and dividing by n, we get what we need

$$\sum_i |U_{\alpha i}|^4 \geq \frac{1}{n}. \tag{6.86}$$

The inequality $\overline{P}(\nu_e \to \nu_e) \geq \frac{1}{2}$ is the simplest example in the 2 generation scheme. We further note that the equality in the above relation, i.e. $\sum_i |U_{\alpha i}|^4 = \frac{1}{n}$ is realized when the equality in the (6.84) is met: $|U_{\alpha 1}|^2 = |U_{\alpha 2}|^2 = ... = |U_{\alpha n}|^2$, namely in the case of "maximal mixing".

6.4 Resonant Neutrino Matter Oscillation

We have seen in the previous section that the time-averaged survival probability never gets lower than $1/n$ (n : the number of generations). Thus, in the 3 generation scheme it is impossible for the vacuum oscillation (neutrino oscillation in the vacuum) to account for data which indicates the survival probability lower than $1/3$, such as the one of the pioneering solar neutrino experiment by R. Davis and collaborators. Furthermore to realize the minimum value $1/3$, a fine tuning of the mixing angles is necessary so that $|U_{\alpha 1}|^2 = |U_{\alpha 2}|^2 = |U_{\alpha 3}|^2$ is realized.

Such difficulty may be overcome, once we invoke to a resonant neutrino oscillation due to the interaction of neutrinos with matter inside the medium, e.g. the sun. Before discussing the detail of the "resonant matter oscillation" it will be useful to note that the matter effect (the interaction of neutrinos with matter) leads to an additional potential energy $V(x)$ only for ν_e. The $V(x)$ then causes the change of the frequency of the matter wave of ν_e, depending on the coordinate x, and therefore time t ($x \simeq t$), for a highly relativistic light neutrinos, while the frequency of the matter wave of another flavor, e.g. ν_μ in 2 generation scheme, stays a constant. Thus it is quite possible that at some point, or equivalently at some time, the frequencies of these two matter waves just coincide, and a resonance phenomenon occurs. If we start from the matter wave of ν_e, at the resonance point almost complete transition of the matter wave from ν_e to ν_μ is possible, even for small mixing angle θ, as far as some suitable condition is met. It is analogous to the case of two tuning-folks of musical instrument, the length of one of these two folks being variable. When the lengths of two folks coincide the vibration of one folk may be resonantly transferred into that of another folk, almost completely. One important thing is that for the transfer to be efficient the variation of the length should be slow enough, i.e. an adiabaticity condition should be met. The role of the mixing angle θ is played by the air (!), which mediates the sound connecting the vibrations of the two tuning folks (see Fig. 6.3).

Now let us confirm the above intuitive argument really holds by use of mathematical formulae. First let us calculate the potential $V(x)$. Since neutrino oscillation is caused by the interference effect (beat) of coherent matter waves with a fixed direction of momentum $\vec{p}$, the potential $V(x)$ should be attributed to the elastic forward scattering of neutrinos off electron, proton and neutron inside the medium, due to the weak interaction. $V(x)$ changes the dispersion relation between $|\vec{p}|$ and E of neutrinos, and therefore the index of refraction of neutrinos. Such elas-

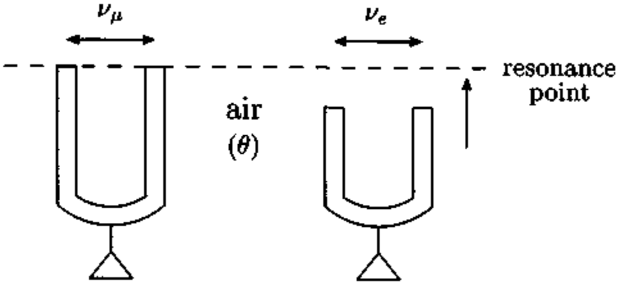


Fig. 6.3

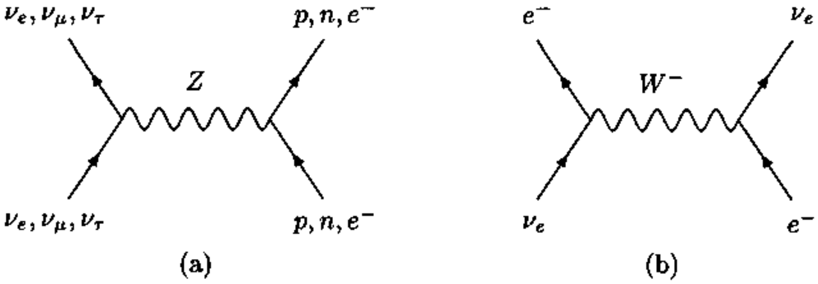


Fig. 6.4

tic scattering with protons and neutrons are possible only through neutral current processes (Fig. 6.4(a)). In the case of neutrino interaction with electron, however, in addition to the neutral current process, only ν_e has a charged current process (Fig. 6.4(b)). The matter effect due to the neutral current process via Z-exchange, $V_n(x)$ is universal for all of three neutrinos. This $V_n(t)$ provides an overall phase factor $\exp(-i\int^t V_n(t')dt')$ for all neutrinos. The universal phase has no physical consequence, since neutrino oscillation is caused by superposition of multiple matter waves with slightly different frequencies, i.e. by the "beat" of matter waves. The remaining charged current process only for ν_e seems to be a scattering process, rather than providing a potential. It, however, turns out that it can be interpreted as a process to give the potential. The charged current process, Fig. 6.4(b), provides an effective 4-Fermi contact interaction Hamiltonian (for neutrino energies $E \ll M_W$)

$$\frac{G_F}{\sqrt{2}} \cdot \overline{\nu_e}\gamma_\mu(1-\gamma_5)e \cdot \bar{e}\gamma^\mu(1-\gamma_5)\nu_e. \tag{6.87}$$

The $(V-A)\times(V-A)$ type 4-Fermi coupling is known to be rewritten by use of Fierz transformation as

$$\frac{G_F}{\sqrt{2}} \cdot \overline{\nu_e}\gamma_\mu(1-\gamma_5)\nu_e \cdot \bar{e}\gamma^\mu(1-\gamma_5)e, \tag{6.88}$$

which is practically equivalent to the neutral current process. Let us recall that in QED, a background electric Coulomb potential $V(x)$ interact with electron as $\bar{e}\gamma_0 e \times V(x)$. Similarly, the above 4-Fermi interaction may be regarded as an interaction of the left-handed ν_e with "Coulomb" type "static" potential $V_c(x)$

$$\overline{\nu_e L}\gamma_0 \nu_{eL} \cdot V_c(x), \tag{6.89}$$

$$V_c(x) = \sqrt{2} G_F N_e(x), \tag{6.90}$$

where $\bar{e}\gamma^0 e$ is identified with the electron number density $N_e(x)$, and other components, $\bar{e}\gamma^i e$ $(i = 1, 2, 3), \bar{e}\gamma^\mu\gamma_5 e$ have been neglected as they are expected to be proportional to the expectation values of velocity or spin of the electron and may be neglected for static and un-polarized medium of electron. Let us again recall that under the influence of 4-potential $A_\mu = (V, \vec{A})$ the dispersion relation between momentum and energy of ν_e is modified into $E - V_c = \sqrt{(\vec{p} - \vec{A})^2 + m^2}$ or $E = \sqrt{(\vec{p} - \vec{A})^2 + m^2} + V_c$. Thus concerning the static un-polarized medium, only ν_{eL} gets additional potential energy V_c, and corresponding change of the energy. Thus the time evolution equation, "Schrödinger equation" in the base of $(\nu_e, \nu_\mu, \nu_\tau)$, (6.65) is accordingly modified into

$$i\frac{d}{dt}\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \left\{ U \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Delta m_{21}^2}{2E} & 0 \\ 0 & 0 & \frac{\Delta m_{31}^2}{2E} \end{pmatrix} \cdot U^\dagger + \begin{pmatrix} a(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} \cdot \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}, \tag{6.91}$$

where the matter effect has been indicated as $a(x) = \sqrt{2} G_F N_e(x)$, instead of V_c, and the replacement $m_i^2 \to \Delta m_{i1}^2$ has been done ignoring an overall phase associated with $m_1^2/2E$.

Though we should treat the realistic 3 generation scheme, such treatment is complicated, generally speaking. As we will discuss in the next section, fortunately there is an approximate reduction formula, valid under the hierarchy of mass-squared differences $\Delta m_{21}^2 \ll \Delta m_{31}^2$, which enables us to reduce the problem in 3 generation scheme to that in an effective 2 generation scheme. Therefore, we will deal with the matter oscillation in 2 generation scheme with a set of generic parameters of one mass-squared difference and one mixing angle $(\Delta m^2, \theta)$, in this section. Such argument will be useful, even in the realistic 3 generation scheme, since the reduction formula is valid.

In the simplified 2 generation scheme, by use of (the simplified version of) (6.91) and (6.68) the time-evolution equation can be explicitly written (after subtracting a common factor $\frac{\Delta m_{21}^2}{2E}\sin^2\theta$ at the diagonal elements) as

$$i\frac{d}{dt}\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \sqrt{2} G_F N_e(t) & \frac{\Delta m^2}{4E}\sin 2\theta \\ \frac{\Delta m^2}{4E}\sin 2\theta & \frac{\Delta m^2}{2E}\cos 2\theta \end{pmatrix} \cdot \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}. \tag{6.92}$$

Since the "Hamiltonian" $H(t)$, the 2×2 matrix, is time dependent, the above differential equation can not be solved analytically, unless the $N_e(t)$ has some specific time dependence. However, in an appropriate circumstance, we may still get a simple analytic formula for the neutrino oscillation. Namely, as we have seen in the analogous case of the tuning-folk, as far as the time dependence of a time variant frequency is mild (the exact meaning of this will be defined shortly), almost complete conversion of one matter wave to the another should be possible. The condition of the "mild time dependence" is called "adiabaticity". The resonance phenomenon, which occurs when the two frequencies become identical, should happen in this system of two neutrinos when the two diagonal elements of $H(t)$ becomes the same, namely at the point where the following condition is met,

$$\sqrt{2}G_F N_e(t) = \frac{\Delta m^2}{2E}\cos 2\theta \equiv \sqrt{2}G_F N_{er}. \tag{6.93}$$

To see these things explicitly, we will move to a (time-dependent) base of neutrino states, where the Hamiltonian $H(t)$ is diagonalized by a time-dependent unitary transformation:

$$U_m(t)^\dagger H(t) U_m(t) = \mathrm{diag}(E_1(t), E_2(t)), \tag{6.94}$$

$$U_m(t) = \begin{pmatrix} \cos\theta_m(t) & \sin\theta_m(t) \\ -\sin\theta_m(t) & \cos\theta_m(t) \end{pmatrix} \tag{6.95}$$

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = U_m(t) \cdot \begin{pmatrix} \nu_{m1} \\ \nu_{m2} \end{pmatrix}, \tag{6.96}$$

where the time-dependent energy eigenvalues and mixing angle are given by

$$E_{1,2}(t) = \frac{1}{2}\left(\sqrt{2}G_F N_e(t) + \frac{\Delta m^2}{2E}\cos 2\theta \right. \\ \left. \pm \sqrt{(\sqrt{2}G_F N_e(t) - \frac{\Delta m^2}{2E}\cos 2\theta)^2 + (\frac{\Delta m^2}{2E}\sin 2\theta)^2}\right), \tag{6.97}$$

$$\tan 2\theta_m = \frac{\frac{\Delta m^2}{2E}\sin 2\theta}{\frac{\Delta m^2}{2E}\cos 2\theta - \sqrt{2}G_F N_e(t)}. \tag{6.98}$$

Fig. 6.5 shows how the mixing angle inside the matter θ_m changes as N_e varies.

We see that the resonance point, satisfying $N_e(t) = N_{er}$, is the point where the conversion of matter wave is most efficient and therefore the mixing θ_m is maximal, i.e.

$$\theta_m = \frac{\pi}{4}. \tag{6.99}$$

We may also say that the θ_m changes most rapidly around the resonance point, and the "resonance region" can be defined as to be a region where $|\tan 2\theta_m| \geq 1$.

Actually, although we have diagonalized $H(t)$, the evolution equation is still a coupled differential equation, since the unitary transformation by $U_m(t)$ is time-dependent. That is why we cannot solve the differential equation analytically.

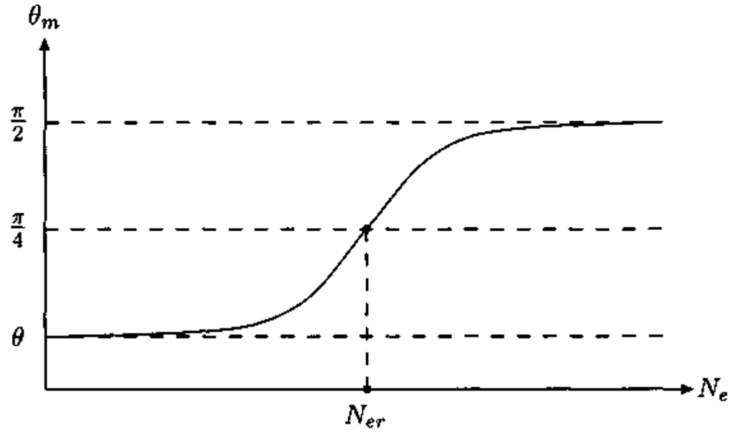


Fig. 6.5

Namely, we get a residual term with off-diagonal elements in the evolution equation in the base of $(\nu_{m1}, \nu_{m2})^t$,

$$i\frac{d}{dt}\begin{pmatrix} \nu_{m1} \\ \nu_{m2} \end{pmatrix} = \left\{ \begin{pmatrix} E_1(t) & 0 \\ 0 & E_2(t) \end{pmatrix} + \begin{pmatrix} 0 & i\dot{\theta}_m \\ -i\dot{\theta}_m & 0 \end{pmatrix} \right\} \cdot \begin{pmatrix} \nu_{m1} \\ \nu_{m2} \end{pmatrix}. \tag{6.100}$$

Then what is a merit of working in this base? Well, in this base the time evolution in the adiabatic case, the case where the adiabaticity condition is met, is easily solved. In fact if the condition

$$|\dot{\theta}_m| \ll |E_2 - E_1| \tag{6.101}$$

is always satisfied, the residual term in the Hamiltonian can be safely ignored, and the system is approximately diagonalized:

$$i\frac{d}{dt}\begin{pmatrix} \nu_{m1} \\ \nu_{m2} \end{pmatrix} \simeq \begin{pmatrix} E_1(t) & 0 \\ 0 & E_2(t) \end{pmatrix} \cdot \begin{pmatrix} \nu_{m1} \\ \nu_{m2} \end{pmatrix}. \tag{6.102}$$

The adiabaticity condition will be the most non-trivial at the resonance point, where the change of θ_m is maximal and the difference of energy eigenvalues is minimum, $|E_2 - E_1| = \frac{\Delta m^2}{2E}\sin 2\theta$. Thus the adiabaticity condition may be written as

$$\frac{\tan 2\theta}{\frac{d\,\log N_e}{dx}|_{res}} \gg \frac{E}{\Delta m^2\,\sin 2\theta}, \tag{6.103}$$

where $\dot{\theta}_m$ has been written in terms of $\frac{d\,\log N_e}{dx}$, which should be evaluated at the resonance point. The physical meaning of this adiabaticity condition is that the wave length of the "beat" at the resonance point, i.e. $\sim 1/|E_2 - E_1| = E/(\Delta m^2 \sin 2\theta)$, should be much smaller than the width of the resonance region, i.e. $(\frac{\Delta m^2}{E}\sin 2\theta)/(\sqrt{2}G_F\frac{dN_e}{dx}|_{res}) = \tan 2\theta/(\frac{d\,\log N_e}{dx}|_{res})$.

The above differential equation is easily solved to yield

$$\nu_{m1}(t) = \exp(-i\int_0^t E_1(t')dt') \cdot \nu_{m1}(0), \tag{6.104}$$

$$\nu_{m2}(t) = \exp(-i\int_0^t E_2(t')dt') \cdot \nu_{m2}(0). \tag{6.105}$$

Thus the survival probability is given as

$$\begin{aligned}
& P(\nu_e \to \nu_e) \\
= & \left|\langle\nu_e|\, T\, \exp(-i\int_0^t H(t')dt')|\nu_e\rangle\right|^2 \\
= & \left|\begin{pmatrix} \cos\theta_m(t) & \sin\theta_m(t) \end{pmatrix} \begin{pmatrix} \exp(-i\int_0^t E_1(t')dt') & 0 \\ 0 & \exp(-i\int_0^t E_2(t')dt') \end{pmatrix}\right. \\
\cdot & \left.\begin{pmatrix} \cos\theta_m(0) \\ \sin\theta_m(0) \end{pmatrix}\right|^2 \\
= & \left|\cos\theta_m(t)\cos\theta_m(0)\, \exp(-i\int_0^t E_1(t')dt')\right. \\
+ & \left.\sin\theta_m(t)\sin\theta_m(0)\, \exp(-i\int_0^t E_2(t')dt')\right|^2 .
\end{aligned} \tag{6.106}$$

If an average is taken over the detection time t, the interference term may be ignored and the formula simplifies into

$$\overline{P}(\nu_e \to \nu_e) = \cos^2\theta_m(t)\cos^2\theta_m(0) + \sin^2\theta_m(t)\sin^2\theta_m(0). \tag{6.107}$$

If the matter effect is ignored, i.e. if $N_e = 0$ and $\theta_m(t) = \theta_m(0) = \theta$, this reduces to (6.78) for the vacuum oscillation.

We now realize that even if the mixing θ is small, almost complete conversion of a solar neutrino, starting as ν_e, into ν_μ at the solar surface is possible, so called MSW effect (Mikheyev and Smirnov, 1985; Wolfenstein, 1978). Namely, setting $\theta_m(t) \simeq \theta$, $\theta_m(0) \simeq \frac{\pi}{2}$ and assuming $N_e(0) \gg N_{er}$, the above formula is simplified into

$$\overline{P}(\nu_e \to \nu_e) \simeq \sin^2\theta, \tag{6.108}$$

which can be sufficiently small even for small mixing θ, in clear contrast to the case of (6.78), due to the resonance phenomenon. Such effective adiabatic conversion mechanism between two states, having level crossing, is visualized in Fig. 6.6.

In Fig. 6.6, the dashed lines indicate the behaviors of the two diagonal elements of $H(t)$ as the functions of N_e. The solid curves correspond to the two eigenvalues $E_1(t)$ and $E_2(t)$. If ν_e is produced at a point with $N_e(0) \gg N_{er}$, ν_e state is approximately on the upper solid curve, as the state ν_e is approximately an eigenstate of $H(0)$ with dominant energy due to the matter effect. As the matter density and

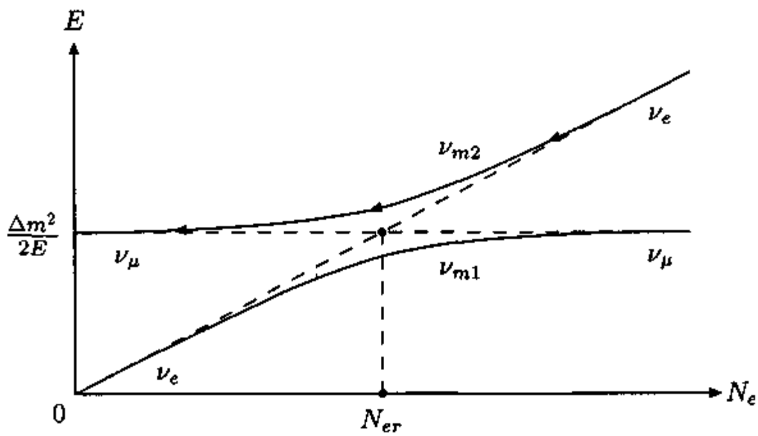


Fig. 6.6

therefore N_e decreases, e.g. as ν_e traverses from the center toward the surface of the sun, the state evolves along the upper solid curve, as long as the adiabaticity is met, and after passing through the resonance (level crossing) point, it will follow the upper dashed line, to end up as the ν_μ state.

In the case that the adiabaticity condition is not satisfied, the above formula should be modified. In this case at the resonance point or at the level crossing, there should be a "jumping" between ν_{m1} and ν_{m2} states, i.e. between the two solid curves. Let the probability of the jumping be P_{jump}. Then the two states interchange by a probability P_{jump} and remain to be the same by a probability $1 - P_{jump}$. Now the survival probability reads as

$$\begin{aligned} \overline{P}(\nu_e \to \nu_e) &= (1 - P_{jump})(\cos^2\theta_m(t)\cos^2\theta_m(0) + \sin^2\theta_m(t)\sin^2\theta_m(0)) \\ &+ P_{jump}(\sin^2\theta_m(t)\cos^2\theta_m(0) + \cos^2\theta_m(t)\sin^2\theta_m(0)) \\ &= \frac{1}{2} + (\frac{1}{2} - P_{jump})\cos 2\theta_m(t)\cos 2\theta_m(0). \end{aligned} \tag{6.109}$$

Setting $\theta_m(t) \simeq \theta,\ \theta_m(0) \simeq \frac{\pi}{2}$, the formula reads as

$$\overline{P}(\nu_e \to \nu_e) \simeq \sin^2\theta + P_{jump}\cos^2\theta. \tag{6.110}$$

This is based on the reasonable assumption that the jumping takes place only near the resonance point. Then it will be a meaningful way of doing to replace the matter effect by its linear approximation so that they coincide at the resonance point. Such a replacement makes it possible to use the Landau-Zener formula for a level crossing (Landau, 1932; Zener, 1932), leading to

$$P_{jump} = \exp(-\frac{\pi\Delta m^2\sin^2 2\theta}{4E\cos 2\theta\ \frac{d\log n_e}{dx}|_{rs}}), \tag{6.111}$$

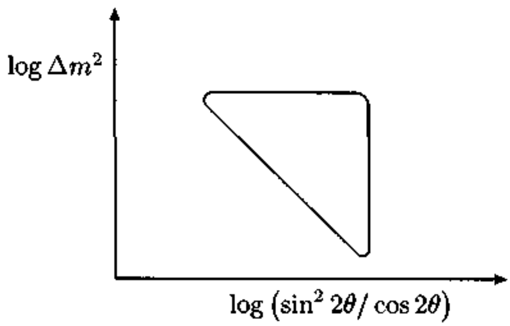


Fig. 6.7

where the derivative $\frac{d \log n_e}{dx}$ should be taken at the resonance point.

In this way, substantial conversion of ν_e is possible by the MSW effect for solar neutrinos, as the typical example of the resonant matter oscillation. When the survival probability $\bar{P}(\nu_e \to \nu_e)$ is fixed by a experimental data, it will determine a curve in the 2-dimensional space of the parameters, $(\frac{\sin^2 2\theta}{\cos 2\theta}, \Delta m^2)$. When $\bar{P}(\nu_e \to \nu_e) \leq 1/2$, the curve is known to draw a closed triangle (roughly speaking) in the log-log plot of the parameters shown in Fig.6.7, which we call MSW triangle.

The three sides of the triangle have their own physical meaning. Namely, the horizontal size represents the parameters, for which the level crossing starts to occur, $N_e(0) \simeq N_{er} = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2} G_F E}$ with E being an average neutrino energy. For $E = 10(MeV)$ and $N_e(0) = 100(g/cc)$, $\Delta m^2 \sim 10^{-5}(eV^2)$. The vertical line is given by a relation $\sin^2\theta$ = observed survival probability, i.e. by use of (6.108). For the parameters on this line, all neutrinos with different energies experience the level-crossing and satisfy the adiabaticity condition. The diagonal line is given by a relation P_{jump} = constant, i.e. by $\Delta m^2 \cdot \frac{\sin^2 2\theta}{4E\cos 2\theta}$ = constant. This line separates the adiabatic (upper) and non-adiabatic regions, since the factor $\Delta m^2 \cdot \frac{\sin^2 2\theta}{4E\cos 2\theta}$ inside P_{jump} determines the adiabaticity of the transition. This MSW triangle plays an important role when we derive the allowed parameter region from the data on solar neutrinos, as we will see briefly later in this Chapter.

6.5 Neutrino Oscillation in the Three Generation Scheme

So far the concrete formulae for the probabilities of neutrino oscillations by use of the mixing angle and the mass-squared difference have been given only for the simplified two generation scheme, for illustrative purpose. Needless to say, to make the analysis realistic, we have to derive formulae in the full three generation scheme, in terms of two mass-squared differences of neutrinos,

$$\Delta m^2_{21}, \quad \Delta m^2_{31}, \tag{6.112}$$

and three mixing angles and one CP violating phase,

$$\theta_{12}, \quad \theta_{23}, \quad \theta_{13}, \quad \delta, \tag{6.113}$$

which appear in the MNS matrix as follows,

$$\begin{aligned}
U &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&\equiv V_{23}V_{13}V_{12},
\end{aligned} \tag{6.114}$$

where s_{ij} and c_{ij} represent $\sin\theta_{ij}$ and $\cos\theta_{ij}$, respectively. (This parameterization just corresponds to (4.183) for the KM matrix in the quark sector.) Such formulae, however, are complicated even for the case of vacuum neutrino oscillations, being described by the 6 parameters $\Delta m^2_{21}, \Delta m^2_{31}, \theta_{12}, \theta_{23}, \theta_{13}, \delta$. Thus given data on the neutrino oscillations provide 5-dimensional hyper-surfaces in the parameter space, which of course cannot be shown graphically. Even if we get the allowed values of the 6 parameters numerically, it may be practically impossible to get some meaningful physical information on the parameters from them. To be worth, we will not be sure with the obtained numerical values of the parameters, which mass-squared difference and mixing angle are really handling the oscillation, while each experimental data on the neutrino oscillation is usually provided in the form of the allowed region of the parameters assuming a simplified two generation scheme with generic parameters $(\theta, \Delta m^2)$.

Fortunately, the hierarchical structure of two mass-squared differences of neutrinos, suggested by the data of atmospheric and solar neutrinos experiments,

$$\Delta m^2_{21} \ll \Delta m^2_{31}, \tag{6.115}$$

enables us to reduce the problem of analyzing neutrino oscillations in the full three generation scheme to that in the effective two generation scheme. Namely, "reduction formulae" are known to exist for each type of neutrino oscillations, which obviously make the analysis of the allowed region of theory parameters quite similar to those in the simplified two generation scheme, thus making the analysis quite transparent and physically meaningful. In addition, in this way we can clearly see which mixing angle and mass-squared difference out of the 6 parameters should be identified with those appearing in the experimental data. As we will briefly discuss in the next section, to explain the data on the atmospheric and solar neutrino oscillation two different scales of mass-squared difference are inevitable: $\Delta m^2_{atm} \simeq 2 \times 10^{-3}$ (eV^2) and $\Delta m^2_{solar} \simeq 7 \times 10^{-5}$ (eV^2). Thus the identification

$$\Delta m^2_{31} = \Delta m^2_{atm} \simeq 2 \times 10^{-3} \ (eV^2),$$

$$\Delta m_{21}^2 = \Delta m_{solar}^2 \simeq 7 \times 10^{-5}\ (eV^2), \tag{6.116}$$

should be made, which means the hierarchy (6.115) is a reasonably good assumption. (In fact numerical calculations show that the reduction formulae given below provide enough accurate oscillation probabilities.)

Now we will discuss these reduction formulae for each of vacuum and resonant matter oscillations, successively below.

6.5.1 *Vacuum Neutrino Oscillation in the Three Generation Scheme*

Even under the hierarchical mass-squared difference (6.115), the reduction formulae of the oscillation probabilities will take different formulae, depending on which of Δm_{21}^2 and Δm_{31}^2 is responsible for the oscillatory time-dependence of the oscillation probability.

(A) Vacuum oscillation due to Δm_{31}^2

We first consider the vacuum neutrino oscillation, where larger mass-squared difference Δm_{31}^2 handles the oscillation. This, in turn means that Δm_{21}^2 is too small to participate to the oscillation. It will be relevant for the experimental situation satisfying,

$$\begin{aligned} \frac{\Delta m_{21}^2}{E}L &= 3.6 \times 10^{-2} \cdot \frac{(\frac{\Delta m_{21}^2}{7\times 10^{-5} eV^2})}{(\frac{E}{1GeV})}(\frac{L}{100km}) \ll 1, \\ \frac{\Delta m_{31}^2}{E}L &= 1.0 \cdot \frac{(\frac{\Delta m_{31}^2}{2\times 10^{-3} eV^2})}{(\frac{E}{1GeV})}(\frac{L}{100km}) \geq 1, \end{aligned} \tag{6.117}$$

where L denotes the distance between the production and the detection points of neutrinos. The oscillation of atmospheric neutrinos and terrestrial "long-baseline" accelerator neutrino oscillation experiments may be classified into this category. It is worth while noticing that the oscillation length of atmospheric neutrino oscillation, implied by the zenith-angle dependence of its survival probability, is not the order of the size of the earth, but is of the order of a few hundred kilometers, roughly the distance from the Super Kamiokande detector till the top of the atmosphere at the Zenith-angle $= \frac{\pi}{2}$. Strictly, for the up-going neutrinos in the Super-Kamiokande detector, i.e. from the opposite side of the earth, the oscillation due to Δm_{12}^2 may not be negligible, though we do not consider the effect in this textbook. The wavelength is comparable to the baselines of K2K (KEK to Kamiokande) and other proposed long baseline experiments.

Thus, ignoring the smaller mass-squared difference, $\Delta m_{21}^2 = 0$, the general oscillation formula (6.67) reduces into

$$P(\nu_\alpha \to \nu_\beta) = |\delta_{\alpha\beta} - U_{\beta 3}U_{\alpha 3}^*(1 - e^{-i\frac{\Delta m_{31}^2}{2E}t})|^2, \tag{6.118}$$

due to the unitarity $\sum_i U_{\beta i}U^*_{\alpha i} = \delta_{\alpha\beta}$. Several survival and transition oscillation probabilities, relevant, e.g., for the atmospheric neutrinos, now takes simple concrete forms in terms of mixing angles

$$\begin{aligned}
P(\nu_\mu \to \nu_\mu) &= 1 - 4(1-|U_{\mu 3}|^2)|U_{\mu 3}|^2 \sin^2(\frac{\Delta m^2_{31}}{4E}t) \\
&= 1 - 4(1-\sin^2\theta_{23}\cos^2\theta_{13})\sin^2\theta_{23}\cos^2\theta_{13}\ \sin^2(\frac{\Delta m^2_{31}}{4E}t) \\
&\simeq 1 - \sin^2 2\theta_{23}\ \sin^2(\frac{\Delta m^2_{31}}{4E}t), \qquad (6.119) \\
P(\nu_e \to \nu_e) &= 1 - 4(1-|U_{e3}|^2)|U_{e3}|^2 \sin^2(\frac{\Delta m^2_{31}}{4E}t) \\
&= 1 - \sin^2 2\theta_{13}\ \sin^2(\frac{\Delta m^2_{31}}{4E}t) \simeq 1, \qquad (6.120) \\
P(\nu_\mu \to \nu_\tau) &= 4|U_{\mu 3}|^2|U_{\tau 3}|^2 \sin^2(\frac{\Delta m^2_{31}}{4E}t) \\
&= \sin^2 2\theta_{23}\cos^4\theta_{13}\ \sin^2(\frac{\Delta m^2_{31}}{4E}t) \\
&\simeq \sin^2 2\theta_{23}\ \sin^2(\frac{\Delta m^2_{31}}{4E}t), \qquad (6.121) \\
P(\nu_\mu \to \nu_e) &= 4|U_{e3}|^2|U_{\mu 3}|^2 \sin^2(\frac{\Delta m^2_{31}}{4E}t) \\
&= \sin^2 2\theta_{13}\sin^2\theta_{23}\ \sin^2(\frac{\Delta m^2_{31}}{4E}t) \simeq 0, \qquad (6.122)
\end{aligned}$$

where the approximate formulae are for small θ_{13}, reported by e.g. CHOOZ experiment,

$$\sin^2 2\theta_{13} \lesssim 0.2. \qquad (6.123)$$

These formulae imply that, roughly speaking, ν_e does not oscillate and is decoupled from other neutrino species, and only $\nu_\mu \leftrightarrow \nu_\tau$ oscillation is possible. This is what we expect for atmospheric neutrinos. We should also note that the formula for the $\nu_\mu \leftrightarrow \nu_\tau$ oscillation (6.121) is of exactly the same form as the transition probability in the simplified two generation scheme, i.e. as (6.69), though the replacement $\theta \to \theta_{23}$, $\Delta m^2_{21} \to \Delta m^2_{31}$ has been made. Namely, in this case the mixing handling the oscillation is known to be θ_{23}. Thus, the experimental data on the atmospheric neutrinos should provide constraints on the set of parameters $(\theta_{23}, \Delta m^2_{31})$, as we will see in the next section. Concerning the possible CP violation in neutrino oscillations, the formulae given above are not suitable, since ignoring Δm^2_{21} and θ_{13} will erase the CP asymmetries, just as in the quark sector (see (9.57), (9.58)).

(B) Vacuum oscillation due to Δm^2_{21}

Next, we consider the vacuum neutrino oscillation, where the smaller mass-squared difference Δm^2_{21} handles the oscillation. This, in turn, means that Δm^2_{31} is so large that the oscillation length due to Δm^2_{31} is much shorter than the baseline L, i.e. the distance between the production and the detection points of the neutrinos. Thus the oscillatory factor due to Δm^2_{31} should be time-averaged, i.e.

$$\cos(\frac{\Delta m^2_{31}}{2E}t) \to 0, \tag{6.124}$$

which means the interference of the matter waves of ν_3 with those of ν_1 and ν_2 can be ignored. This treatment will be relevant for the experimental situation satisfying,

$$\begin{aligned} \frac{\Delta m^2_{21}}{E}L &= 7.2 \cdot \frac{(\frac{\Delta m^2_{21}}{7\times 10^{-5}eV^2})}{(\frac{E}{5MeV})}(\frac{L}{100km}) \simeq 1, \\ \frac{\Delta m^2_{31}}{E}L &= 2.0 \times 10^2 \cdot \frac{(\frac{\Delta m^2_{31}}{2\times 10^{-3}eV^2})}{(\frac{E}{5MeV})}(\frac{L}{100km}) \gg 1. \end{aligned} \tag{6.125}$$

This is exactly the case for the KamLAND experiment, recently started in Japan.

Ignoring the interference of the matter waves of ν_3 with those of ν_1 and ν_2, the general oscillation formula (6.67) reduces into

$$P(\nu_\alpha \to \nu_\beta) = |\delta_{\alpha\beta} - U_{\beta 3}U^*_{\alpha 3} - U_{\beta 2}U^*_{\alpha 2}(1 - e^{-i\frac{\Delta m^2_{21}}{2E}t})|^2 + |U_{\beta 3}|^2|U_{\alpha 3}|^2. \tag{6.126}$$

In particular, the survival probabilities for $\alpha = \beta$ are expressed by the following reduction formula to the effective two generation scheme:

$$P(\nu_\alpha \to \nu_\alpha) = (1 - |U_{\alpha 3}|^2)^2 \; P_{eff}(\nu_\alpha \to \nu_\alpha) + |U_{\alpha 3}|^4, \tag{6.127}$$

where P_{eff} denotes the survival probability in the effective two generation scheme, which is obtained by the replacing the factor $\sin^2\theta$ in the formula for two generation scheme by a factor $\frac{|U_{\alpha 2}|^2}{1-|U_{\alpha 3}|^2}$. Namely,

$$P_{eff}(\nu_\alpha \to \nu_\alpha) = 1 - 4\frac{|U_{\alpha 2}|^2(1 - |U_{\alpha 2}|^2 - |U_{\alpha 3}|^2)}{(1 - |U_{\alpha 3}|^2)^2} \sin^2(\frac{\Delta m^2_{21}}{4E}t). \tag{6.128}$$

For instance, the concrete reduction formula in terms of mixing angle for the ν_e survival probability, which we denote by S, is given as

$$\begin{aligned} S &= \cos^4\theta_{13} \cdot S_{eff}(\theta_{12}, \Delta m^2_{21}) + \sin^4\theta_{13}, \\ &\simeq \cos 2\theta_{13} \cdot S_{eff}(\theta_{12}, \Delta m^2_{21}) \end{aligned} \tag{6.129}$$

where the survival probability of ν_e in the effective two generation system, denoted by S_{eff} is described by a set of parameters $(\theta_{12}, \Delta m^2_{21})$ and reads as

$$S_{eff}(\theta_{12}, \Delta m^2_{21}) = 1 - \sin^2 2\theta_{12} \sin^2(\frac{\Delta m^2_{21}}{4E}t). \tag{6.130}$$

The approximate formula in the second line of (6.129) is for small θ_{13}, (6.123). The formula (6.129) with (6.130) will be also applicable to the survival probability of $\overline{\nu_e}$, relevant for the KamLAND experiment, as well, as long as the small matter effect due to the neutrino interaction with the earth is negligible. Thus, in this case the oscillation is known to be handled by the mixing angle θ_{12}.

(C) CP violation in neutrino oscillation

As we will discuss in some detail for the quark sector in Chapter 9, to get CP violation we need the full interplay of three distinct generations of quark. In other words, if there is mass degeneracy in up- or down-type quark sector, or if some of generation mixing angles disappear, the CP violating observables will vanish (see (9.57) and (9.58)). The same arguments hold for the lepton sector (at least when neutrinos have Dirac masses). Thus the formulae given in (A) and (B) above, where either Δm^2_{31} or Δm^2_{21} is responsible for the oscillations, are not applicable when we consider the CP violation in neutrino oscillation, and we should go back to the original formula (6.67).

The clear indication of CP violation in the leptonic sector will be the asymmetry between the oscillation probabilities of neutrinos and the corresponding anti-neutrinos defined by

$$A^{CP}_{\alpha\beta} = P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta). \tag{6.131}$$

The consequence of the CPT theorem shown in (6.72) (for the negligible matter effect) implies that the CP asymmetry is identical to the T asymmetry defined by

$$A^{T}_{\alpha\beta} = P(\nu_\alpha \to \nu_\beta) - P(\nu_\beta \to \nu_\alpha). \tag{6.132}$$

Namely, $A^{CP}_{\alpha\beta} = A^{T}_{\alpha\beta}$. This can be confirmed explicitly in the formula (6.67). Namely, the charge conjugation causes the replacement $U_{\alpha i} \to U^*_{\alpha i}$ for every MNS matrix elements, which is equivalent to the replacement $\alpha \leftrightarrow \beta$ in (6.67).

Then, a trivial relation $A^{T}_{\alpha\beta} = -A^{T}_{\beta\alpha}$ tells us

$$A^{CP}_{\alpha\beta} = -A^{CP}_{\beta\alpha}, \tag{6.133}$$

which in particular means $A^{CP}_{\alpha\alpha} = 0$, namely

$$P(\nu_\alpha \to \nu_\alpha) = P(\bar{\nu}_\alpha \to \bar{\nu}_\alpha). \tag{6.134}$$

Thus the CP violating effects do not appear in the survival probabilities and appear only in the transition probabilities. On the other hand, since the neutrino oscillation we are considering here does not break lepton number, the unitarity, or the probability conservation read as

$$\sum_{\beta=e,\mu,\tau} P(\nu_\alpha \to \nu_\beta) = \sum_{\beta=e,\mu,\tau} P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = 1, \tag{6.135}$$

which means $\sum_\beta A^{CP}_{\alpha\beta} = 0$. Together with $A^{CP}_{\alpha\alpha} = 0$ and $A^{CP}_{\alpha\beta} = -A^{CP}_{\beta\alpha}$ we get a relation for $\alpha = e$, $A^{CP}_{e\mu} = A^{CP}_{\tau e}$. Similarly, for $\alpha = \mu$ we get $A^{CP}_{\mu\tau} = A^{CP}_{e\mu}$. We thus obtain a remarkable relation as the specific feature of the three generation model,

$$A^{CP}_{e\mu} = A^{CP}_{\mu\tau} = A^{CP}_{\tau e} \equiv A^{CP}, \tag{6.136}$$

This is essentially the consequence of the fact that in the three generation model the CP violating phase is unique, i.e. δ alone.

Let us now derive the explicit form of A^{CP}. From (6.67) we realize that for an arbitrary choice of the pair (α, β),

$$A^{CP}_{\alpha\beta} = P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = -4\sum_{i<j} \mathrm{Im}(U_{\alpha i}U^*_{\beta i}U_{\beta j}U^*_{\alpha j}) \cdot \sin(\frac{\Delta m^2_{ij}}{2E}t). \tag{6.137}$$

Write $\mathrm{Im}(U_{\alpha i}U^*_{\beta i}U_{\beta j}U^*_{\alpha j}) \equiv J_{\alpha\beta,ij}$. By definition we readily know $J_{\alpha\beta,ij} = -J_{\beta,\alpha,ij}$ and $J_{\alpha\beta,ji} = -J_{\alpha\beta,ij}$. On the other hand, the orthogonality of the MNS matrix implies $\sum_i J_{\alpha\beta,ij} = 0$. We thus know for $j = 2$, e.g., $J_{\alpha\beta,12} + J_{\alpha\beta,22} + J_{\alpha\beta,32} = J_{\alpha\beta,12} + J_{\alpha\beta,32} = 0$, Similarly, for $j = 3$ we get $J_{\alpha\beta,13} + J_{\alpha\beta,23} = 0$. By use of the property $J_{\alpha\beta,ji} = -J_{\alpha\beta,ij}$, we thus conclude $J_{\alpha\beta,12} = J_{\alpha\beta,23} = J_{\alpha\beta,31}$. Similar reasoning provides $J_{e\mu,ij} = J_{\mu\tau,ij} = J_{\tau e,ij}$. In this way we have demonstrated that actually $J_{\alpha\beta,ij}$ is unique, in spite of the possible choices of the combinations of α, β and i, j. Hence we define $J \equiv J_{e\mu,12}$, which is the leptonic counterpart of the Jarlskog parameter (9.58) in the quark sector. In terms of the parameterization in (6.114),

$$J = c_{12}s_{12}c_{23}s_{23}c^2_{13}s_{13}s_\delta, , \tag{6.138}$$

where $s_\delta = \sin\delta$ with δ being the CP violating phase. Now setting as $\alpha = e$, $\beta = \mu$ in (6.137) and utilizing $J_{e\mu,12} = J_{e\mu,23} = J_{e\mu,31}$, we readily get

$$A^{CP} = -4J\ \{\sin(\frac{\Delta m^2_{12}}{2E}t) + \sin(\frac{\Delta m^2_{23}}{2E}t) + \sin(\frac{\Delta m^2_{31}}{2E}t)\}, \tag{6.139}$$

where $\Delta m^2_{ij} = m^2_i - m^2_j$. The appearance of the factor J is quite reasonable, as it should be the unique re-phasing invariant measure of CP violation in the three generation model, as we will see in section 9.4.

The above formula clearly shows that, as we anticipated from the introductory argument, the CP asymmetry A^{CP} vanishes when (at least) one of the following situation is realized, though they may be unrealistic: (1) there is a degeneracy of neutrino masses, i.e. $\Delta m^2_{21} = 0$ or $\Delta m^2_{31} = 0$, (2) one of the mixing angles θ_{ij} or CP violating phase δ vanishes, (3) time average of the neutrino oscillation probabilities, i.e. the average of the factor $\sin(\frac{\Delta m^2_{ij}}{2E}t)$ is taken. Any experiment, which aims at the observation of the CP asymmetry should carefully avoid the situations (1) and (3). In particular, the experiment should be sensitive to the smaller mass-squared difference Δm^2_{21}. It should also be mentioned that the presence of the matter effect mimic the CP asymmetry, just because the matter effect for neutrinos and that for anti-neutrinos are not the same. Thus how to avoid or extract the matter effect is another challenging issue.

6.5.2 *Resonant Matter Oscillations of Neutrinos in the Three Generation Scheme*

Once the matter effect due to the weak interaction of neutrinos with the matter, e.g. inside the sun, is included, the time evolution equation is not analytically solvable, and the analysis of the oscillation probabilities in the full three generation scheme becomes quite cumbersome, in general. Still, however, some reduction formulae are possible if the matter effect is much smaller than Δm^2_{31}, i.e.

$$\frac{\Delta m^2_{21}}{2E},\ \sqrt{2}G_F N_e \ll \frac{\Delta m^2_{31}}{2E}, \tag{6.140}$$

which is the case for solar neutrinos. (Concerning the neutrinos emitted from supernovae, the matter effect can easily exceed $\Delta m^2_{31}/(2E)$ in the region near to the core, and two-step level crossings become possible. Though there exists an extension of the reduction formula, applicable for the case, it is out of scope of this textbook.) Here let us focus on the n_e survival probability S. We will now see that, even in the presence of the matter effect, a similar reduction formula to the one given in (6.129) holds;

$$S = \cos^4\theta_{13} \cdot S_{eff}(\theta_{12}, \Delta m^2_{21}; a_{eff}) + \sin^4\theta_{13}, \tag{6.141}$$

where $a_{eff} \equiv \cos^2\theta_{13}\ a(x) = \cos^2\theta_{13}\ \sqrt{2}G_F N_e(x)$ is the "effective" matter effect, reduced by the factor $\cos^2\theta_{13}$, in the effective two generation scheme (Lim, 1987; Smirnov, 1992; Shi and Schramm, 1992).

Now let us derive this formula. We start with the time-evolution equation in the three generation scheme, (6.91). What we attempt is to separate the 3×3 Hamiltonian matrix, though it cannot be exactly diagonalized because of the presence of the matter effect, into those for one and two neutrino systems, i.e. $3 \to 1 + 2$, by making some suitable approximation. Namely, we attempt to make the Hamiltonian of the form of block-diagonal, relying on the hierarchy (6.140). It is reasonable to expect that one mass eigenstate ν_3, having the dominant "energyh, i.e. the diagonal matrix element $\frac{\Delta m^2_{31}}{2E}$, is decoupled form other two states, orthogonal to ν_3, namely the linear combinations of ν_1 and ν_2. Thus, in the base of mass-eigenstates, the Hamiltonian is approximated by $H \simeq \mathrm{diag}(0, 0, \frac{\Delta m^2_{31}}{2E})$, roughly speaking. Instead of mass-eigenstates ν_1 and ν_2, however, it will be convenient to take their linear combination defined by $\nu'_e \equiv \cos\theta_{12}\nu_1 + \sin\theta_{12}\nu_2$, $\nu'_\mu \equiv -\sin\theta_{12}\nu_1 + \cos\theta_{12}\nu_2$. The reason is, in this way, the 2×2 sub-matrix of the Hamiltonian in the space of ν'_e and ν'_μ has a similar form to that in the two generation scheme, since only ν'_e has the matter effect and the term of $a(x)$ appears in the diagonal element of the sub-matrix. Namely, we will work in the base

$$\xi \equiv \begin{pmatrix} \nu_e' \\ \nu_\mu' \\ \nu_3 \end{pmatrix} = \begin{pmatrix} \cos\theta_{12}\nu_1 + \sin\theta_{12}\nu_2 \\ -\sin\theta_{12}\nu_1 + \cos\theta_{12}\nu_2 \\ \nu_3 \end{pmatrix}. \tag{6.142}$$

Recall that from (6.52), (6.114) the relation $\psi_w = V_{23}V_{13}V_{12}\psi_m$ holds between the column vectors of weak- and mass- eigenstates, $\psi_w \equiv (\nu_{eL}, \nu_{\mu L}, \nu_{\tau L})^t$ and $\psi_m \equiv (\nu_{1L}, \nu_{2L}, \nu_{3L})^t$. Thus, the relation $(V_{23}^\dagger \psi_w) = V_{13}(V_{12}\psi_m)$ means $\xi = V_{12}\psi_m = V_{13}^\dagger(V_{23}^\dagger\psi_w)$. More explicitly,

$$\xi = \begin{pmatrix} \nu_e' \\ \nu_\mu' \\ \nu_3 \end{pmatrix} = \begin{pmatrix} \cos\theta_{13}\nu_e - \sin\theta_{13}e^{-i\delta}\tilde{\nu}_\tau \\ \tilde{\nu}_\mu \\ \sin\theta_{13}e^{i\delta}\nu_e + \cos\theta_{13}\tilde{\nu}_\tau \end{pmatrix}, \tag{6.143}$$

where

$$\begin{aligned} \tilde{\nu}_\mu &\equiv \cos\theta_{23}\nu_\mu - \sin\theta_{23}\nu_\tau \simeq \frac{1}{\sqrt{2}}(\nu_\mu - \nu_\tau), \\ \tilde{\nu}_\tau &\equiv \sin\theta_{23}\nu_\mu + \cos\theta_{23}\nu_\tau \simeq \frac{1}{\sqrt{2}}(\nu_\mu + \nu_\tau), \end{aligned} \tag{6.144}$$

where the approximate relations are for $\theta_{23} \simeq \frac{\pi}{4}$, as suggested by the data on the atmospheric neutrino oscillation. Actually, as far as the survival probability of ν_e is concerned, we may set $\theta_{23} = 0$ without changing the answer, since both of initial and final states have nothing to do with ν_μ, ν_τ and the matter effect is invariant under the rotation of these states. As we expected, among ν_e' and ν_μ', only ν_e' has the ν_e component and, therefore, the matter effect.

Now the time-evolution equation in the base of ξ is written as

$$\begin{aligned} i\frac{d\xi}{dt} &= \left\{ V_{12} \begin{pmatrix} 0 & & \\ & \Delta_{12} & \\ & & \Delta_{13} \end{pmatrix} V_{12}^\dagger + V_{13}^\dagger \begin{pmatrix} a(t) & & \\ & 0 & \\ & & 0 \end{pmatrix} V_{13} \right\} \xi \\ &= \left\{ \begin{pmatrix} \Delta_{12}s_{12}^2 + ac_{13}^2 & \Delta_{12}s_{12}c_{12} & 0 \\ \Delta_{12}s_{12}c_{12} & \Delta_{12}c_{12}^2 & 0 \\ 0 & 0 & \Delta_{13} + as_{13}^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & as_{13}c_{13} \\ 0 & 0 & 0 \\ as_{13}c_{13} & 0 & 0 \end{pmatrix} \right\} \xi, \end{aligned} \tag{6.145}$$

where $\Delta_{12} \equiv \frac{\Delta m_{21}^2}{2E}$, $\Delta_{13} \equiv \frac{\Delta m_{31}^2}{2E}$ and $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$. The second matrix in the right-hand-side of the above equation may be treated as a small perturbation under the hierarchy (6.140), and may be ignored. One may wonder why the small quantities in the 2×2 sub-matrix in the first matrix in the right-hand-side are kept. This is because at the leading 0-th order of perturbation, i.e. eliminating these small quantities as well, the Hamiltonian has degenerate eigenvalues 0 in the space of sub-matrix. Thus the two eigenstates in the sub-space is not fixed at the 0-th order, and the small perturbation should be kept only in the sub-space. We can read off the factor $\cos^2\theta_{13}\ a(t)$ in the sub-matrix, which comes from the relation $\nu_e' = \cos\theta_{13}\nu_e - \sin\theta_{13}e^{-i\delta}\bar{\nu}_\tau$. Namely, in this sub-space the matter effect is reduced to

$$a_{eff} = \cos^2\theta_{13}\ a(t). \tag{6.146}$$

Thus the time-evolution is described approximately by

$$i\frac{d\xi}{dt} \simeq \begin{pmatrix} \Delta_{12}s_{12}^2 + ac_{13}^2 & \Delta_{12}s_{12}c_{12} & 0 \\ \Delta_{12}s_{12}c_{12} & \Delta_{12}c_{12}^2 & 0 \\ 0 & 0 & \Delta_{13} + as_{13}^2 \end{pmatrix} \xi. \tag{6.147}$$

Let us note that the separation $3 \to 1+2$ has been realized, and the time-evolution in the subspace of $(\nu_e', \nu_\mu')^t$ is exactly the same as that of the two generation scheme, except for the replacement of the matter effect $a \to a_{eff}$.

Since ν_e state corresponds to $(c_{13}, 0, s_{13}e^{i\delta})^t$, the survival probability S is given as

$$\begin{aligned} S &= |c_{13}^2 A_{eff}(\nu_e' \to \nu_e') + s_{13}^2\ e^{-i(\Delta_{13}+a\ s_{13}^2)t}|^2 \\ &= \cos^4\theta_{13} \cdot S_{eff}(\theta_{12}, \Delta m_{21}^2; a_{eff}) + \sin^4\theta_{13} \\ &\simeq \cos 2\theta_{13} \cdot S_{eff}(\theta_{12}, \Delta m_{21}^2; a_{eff}), \end{aligned} \tag{6.148}$$

which is nothing but the reduction formula (6.141). The approximate formula in the third line is for small θ_{13}, (6.123). In the derivation we have ignored the interference of ν_3 with other two states, as the oscillation due to Δ_{13} is very rapid: $e^{-i\Delta_{13}t} \to 0$. The effective survival probability $S_{eff}(\theta_{12}, \Delta m_{21}^2; a_{eff})$ should be calculated by use of the 2×2 sub-matrix in the Hamiltonian of (6.147). Let us note that if we neglect the small θ_{13}, $S = S_{eff}$ and $\nu_e = \nu_e'$. Namely, in this case the two generation treatment is exact and the solar neutrinos oscillate into $\nu_\mu' = \tilde{\nu}_\mu$;

$$\nu_e \ \to \ \frac{1}{\sqrt{2}}(\nu_\mu - \nu_\tau), \tag{6.149}$$

and we expect roughly the same amount of ν_μ and ν_τ in the solar neutrinos. The error of the reduction formula (6.141) is anticipated to be handled by the relative strength of the ignored off-diagonal matrix element to the dominant matrix element, of the order $\frac{a\ \sin\theta_{13}}{(\frac{\Delta m_{31}^2}{2E})} \leq 10^{-2}$, which has been confirmed by numerical calculation.

6.6 Atmospheric and Solar Neutrino Oscillations

The smallness of neutrino masses and therefore small mass-squared differences means that the oscillation lengths of possible neutrino oscillations are sizable, and to test these oscillations are very challenging. Though a long distance L from the source to the detector of the neutrinos is desirable to detect such oscillations with long wave lengths, it in turn means the neutrino flux rapidly decreases as $1/L^2$.

Thus the neutrino source with large L and very intense flux should be ideal one to test the oscillation. We can find such ideal neutrinos sources with astrophysical or cosmological origin, i.e. atmospheric neutrinos and solar neutrinos.

As we now very briefly discuss, both of these neutrinos have experimental puzzles, called atmospheric and solar neutrino problems (anomalies): the observed event rates of these two-types of neutrinos are significantly smaller than the theoretical predictions. We will see below that these reductions of the neutrino event rates are naturally accounted for by virtue of the atmospheric and solar neutrino oscillations.

6.6.1 *Atmospheric Neutrino Oscillation*

It has been known for long time that high energy particles (mainly protons) are flowing toward the earth, almost isotropically. These particles are called (primary) cosmic ray, and interacts with the atmosphere surrounding the earth via the strong interaction, to produce pions. The decay of the pions due to the weak interaction then copiously produces neutrinos, called "atmospheric neutrinos", through the chain of decay processes:

$$\pi^+ \to \mu^+ + \nu_\mu, \quad \mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu. \tag{6.150}$$

Let us note that the decay rate of $\pi^+ \to e^+ + \nu_e$ is strongly suppressed compared with that of $\pi^+ \to \mu^+ + \nu_\mu$ shown above roughly by a factor $(m_e/m_\mu)^2$ (see (2.49)). Though the absolute values of ν_e and ν_μ fluxes have some uncertainties in their calculations, in the ratio of these fluxes $(\nu_\mu + \bar{\nu}_\mu)/(\nu_e + \bar{\nu}_e)$ (ν_μ denoting the ν_μ flux, etc.), such uncertainty is considerably reduced. The above decay chain suggests that the ratio is roughly 2.

The atmospheric neutrino experiment by Super-Kamiokande collaboration claims that the "double ratioh of the observed $(\nu_\mu + \bar{\nu}_\mu)/(\nu_e + \bar{\nu}_e)$ to that of the predicted value remarkably deviates from 1,

$$R = \frac{(\nu_\mu + \bar{\nu}_\mu)_{obs}/(\nu_e + \bar{\nu}_e)_{obs}}{(\nu_\mu + \bar{\nu}_\mu)_{pred}/(\nu_e + \bar{\nu}_e)_{pred}} \sim 0.6. \tag{6.151}$$

This discrepancy is so-called atmospheric neutrino problem (anomaly). To be more precise, Super-Kamiokande data show that the ν_μ capture rate is considerably smaller that the expectation, while that of ν_e is consistent with the prediction. Thus, probably the most natural explanation of this problem is to invoke to the atmospheric neutrino oscillation

$$\nu_\mu \quad \to \quad \nu_\tau, \tag{6.152}$$

while atmospheric ν_e do not experience any oscillation. This observation is consistent with the theoretical expectations of neutrino oscillation probabilities for small θ_{13} given in (6.119) - (6.122). Comparing the survival probability $P(\nu_\mu \to \nu_\mu)$,

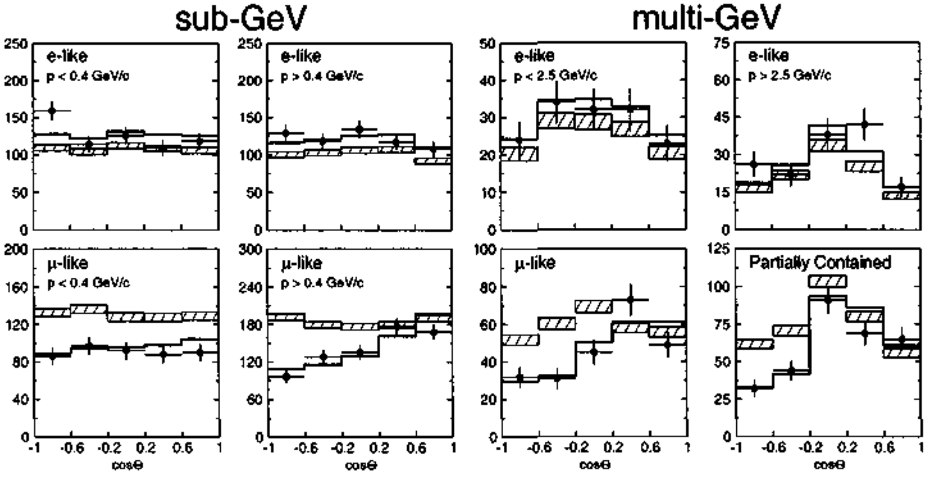


Fig. 6.8 The Zenith-angle (θ) dependence of μ-like and e-like neutrino events for the scattered particles in the energy ranges of sub-GeV and multi-GeV. The hatched regions are expected events for no neutrino oscillation, while the bold lines indicate the (best-fit) expected events assuming the neutrino oscillation $\nu_\mu \to \nu_\tau$. From the result of Super-Kamiokande collaboration, (Super-Kamiokande collaboration, 1998).

(6.119), with the data, Super-Kamiokande has derived the allowed parameter region

$$\Delta m_{31}^2 = (1.3 - 3.0) \times 10^{-3}\ (eV^2), \tag{6.153}$$

$$\sin^2 2\theta_{23} > 0.9. \tag{6.154}$$

To be interesting, the best fit value of $\sin^2 2\theta_{23}$ is 1. Namely the maximal mixing has been realized in the lepton sector, in clear contrast to the case of quark sector !

To get the restrictive range of Δm_{31}^2, the data on the Zenith-angle dependence is very helpful. Namely, the dependence of the atmospheric neutrino capture rate on the Zenith-angle θ at Kamioka, shown in Fig. 6.8, is probably the most convincing direct evidence of neutrino oscillation.

The figures tell us that the muon-like event rates at $\cos\theta = -1$, i.e. $\theta = \pi$, corresponding to the atmospheric neutrinos from the opposite side of the earth, indicates a clear deficit, while the event rate at $\theta = 0$ is quite consistent with the prediction. This may be attributed to the oscillation of atmospheric neutrinos ν_μ, which traverse long way toward the Super-Kamiokande detector. The remarkable decrease of the capture rate at $\theta = \frac{\pi}{2}$ shows the wave length of the oscillation is of the order a few hundred kilometers. This is why the terrestrial test of the atmospheric neutrino oscillation at the long-baseline experiment, such as K2K, is possible.

6.6.2 *Solar Neutrino Oscillation*

Another ideal source of neutrinos to perform the neutrino oscillation experiment ,sensitive to the very small neutrino mass-squared differences, is the Sun. As is well-known, the energy emitted by the Sun is due to the nuclear fusion processes at the center region of the Sun. A main series of reactions is called "p-p chainh:

$$\begin{aligned} p+p &\rightarrow \mathrm{D}+e^{+}+\nu_e, \quad <E_\nu>=0.26(MeV), \\ &\vdots \\ e^{-}+{}^{7}\mathrm{Be} &\rightarrow {}^{7}\mathrm{Li}+\nu_e, \quad E_\nu=0.86(MeV), \\ &\vdots \\ {}^{8}\mathrm{B} &\rightarrow {}^{8}\mathrm{Be}^{*}+e^{+}+\nu_e, \quad <E_\nu>=7.2(MeV), \\ &\vdots \quad , \end{aligned} \tag{6.155}$$

where three reactions, which mainly contribute to the event rates at the on-going solar neutrino experiments, have been shown, together with the average energies $<E>$ (a fixed energy for the second reaction concerning Be) of the emitted neutrinos. Actually, the second and the third reactions actually belong to different branches of the chain reactions. The neutrinos emitted by these three processes are called *pp*, Be and B neutrinos, respectively. The *pp* neutrino dominants the solar neutrino flux, though its average energy is relatively low and its detection is not easy, while Be and B neutrinos have higher energies and are detectable even in the experiments with relatively higher threshold energies, such as Super-Kamiokande, Homestake and SNO experiments, in the mines in Japan, U.S. and Canada, respectively .

The chain reactions are summarized in the following net reaction of the nuclear fusion

$$2e^{-}+4p \rightarrow {}^{4}\mathrm{He}+2\nu_e+\gamma \ (26.73MeV). \tag{6.156}$$

Though of course the nuclear fusion itself is due to the strong interaction, the presence of the weak interaction causes the emission of the solar neutrinos ν_e.

The experiment for the detection of solar neutrinos, solar neutrino experiment, started by the pioneering work by R. Davis and collaborators, at Homestake mine, the "Cl experiment" by use of C_2Cl_4. Now the data on the solar neutrino capture rates are available from four different types of solar neutrino experiments:

1. Cl experiment :
   $\nu_e+{}^{37}\mathrm{Cl} \rightarrow {}^{37}\mathrm{Ar}+e^{-}, \quad obs./exp.=0.34\pm0.03,$
   (mainly) sensitive to B, Be neutrinos ($E_{th}=0.81MeV$)
2. Super-Kamiokande experiment :
   $\nu+e^{-} \rightarrow \nu+e^{-}, \quad obs./exp.=0.47\pm0.02,$

sensitive to B neutrinos ($E_{th} = 5MeV$)

3. Ga experiment (SAGE, GNO(GALLEX)) :
   $\nu_e +^{71}\text{Ga} \rightarrow {}^{71}\text{Ge} + e^-$, $obs./exp. = 0.55 \pm 0.05$,
   (mainly) sensitive to pp, Be, B neutrinos ($E_{th} = 0.23MeV$)
4. SNO experiment :
   (charged current reaction)
   $\nu_e + \text{D} \rightarrow e^- + p + p$, $obs./exp. = 0.35 \pm 0.02$,
   (neutral current reaction)
   $\nu + \text{D} \rightarrow \nu + n + p$, $obs./exp. = 1.01 \pm 0.13$,
   (sensitive to B neutrinos) ($E_{th} = 6.8MeV$), (6.157)

where Cl and Ga experiments are radio-chemical experiments, while Super-Kamiokande and SNO experiments utilize elastic scattering of neutrinos off electron (Super-Kamiokande) and the charged current interaction of ν_e and the neutral current interactions of all species of neutrinos, ν_e, ν_μ, and ν_τ, with the deuteron D (SNO). In the above equations, "obs./exp." means the ratio of the observed solar neutrino capture rate to the expected one, predicted by the standard solar model (Bahcall, et al., 2001). E_{th} denotes the threshold energy of each experiment. We realize that these experiments have all different E_{th} and are sensitive to different energy ranges. Hence we can get independent complementary data from these experiments, which enables us to restrict the allowed region of parameters of the theory, as we will below. We know from the above "obs./exp." ratios that all experiments, except SNO neutral current experiment, have shown clear deficits of solar neutrinos. This anomaly is called "solar neutrino problem".

This solar neutrino problem find a natural solution in the neutrino oscillation of solar ν_e. As the Cl and SNO charged current experiments claim that $obs./exp. \simeq 1/3$, it is difficult to explain the deficits in the framework of the vacuum oscillation. Even though the (time-averaged) survival probability $\bar{P}(\nu_e \rightarrow \nu_e) = 1/3$ is possible in he three scheme, it requires maximal mixings, i.e. the fine tuning of mixing angles (see (6.80) and the argument below that equation). Thus we should rely on the resonant matter oscillation of solar neutrinos in the three generation scheme, discussed in subsection 6.5.2. We have seen there that, solar ν_e, roughly speaking, oscillate into the even mixture of "active" (neutrino states with weak interaction) ν_μ and ν_τ (see (6.149)). Comparing the observed $obs./exp.$ ratio with the reduction formula (6.141) or (6.148) of the survival probability, we get (with a little uncertainty of the small θ_{13}) a MSW triangle, Fig. 6.7, in the parameter space of $(\log(\sin^2 2\theta_{12}/\cos 2\theta_{12}), \log \Delta m^2_{21})$ for each of experiment. Each experiment, having different deficit rates and energy sensitivities, will draw different MSW triangles. By taking the overlap of these triangles, we finally get a allowed region of the parameters, called LMA (Large Mixing Angle)-MSW solution, as is seen in the right

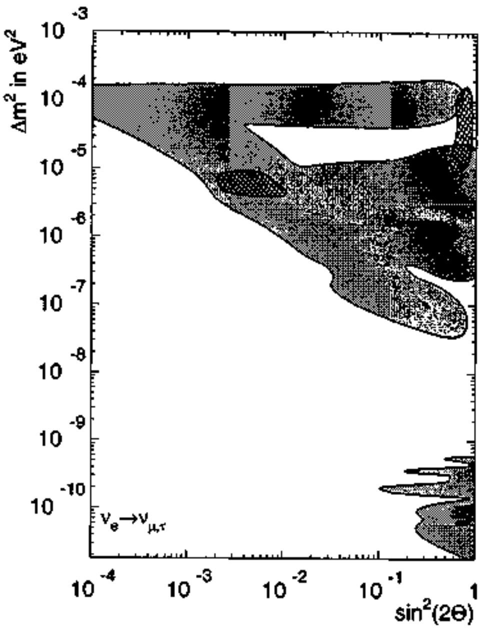


Fig. 6.9 The allowed area in the parameter space of $(\sin^2(2\theta_{12}), \Delta m^2_{12})$, where θ_{12} and Δm^2_{12} are simply written as θ and Δm^2, imposed by the data of solar neutrino experiments. The hatched area is the allowed area at 95% C.L. obtained by the combined analysis on the neutrino flux observed in the Ga, Cl and Super-Kamiokande experiments. The area of the triangular shape, roughly speaking, is the exclusion area due to the day/night spectrum analysis by Super-Kamiokande. From the result of (Super-)Kamiokande collaboration, (Super-Kamiokande collaboration, 2001).

upper area of Fig. 6.9.

The best fit values are:

$$\Delta m^2_{21} = 7.0 \times 10^{-5}\ (eV^2), \tag{6.158}$$

$$\tan^2\theta_{12} = 0.34. \tag{6.159}$$

Let us note the SNO neutral current result $obs./exp. = 1.01 \pm 0.13$ in (6.157) is consistent with the neutrino oscillation (6.149), since the ν_μ and ν_τ are active states and contribute to the capture rates of solar neutrinos, just as ν_e does. Namely only in the SNO neutral current experiment, the capture rate is expected to be the same as the prediction of the standard solar model, which is exactly what has been confirmed by the experiment.

When we discuss the deficit of the solar neutrinos, of course we have to know the prediction of the standard solar model. There exist, however, a way to confirm the neutrino oscillation, irrespectively of the solar model, by use of the combined results of Super-Kamiokande (Super-Kamiokande collaboration, 2001) and SNO charged current reaction (SNO collaboration, 2001)! This should be a direct and convincing proof of the solar neutrino oscillation. The clue is in the fact that in the elastic scattering of solar neutrinos off electron, not only ν_e but also ν_μ and ν_τ contribute to the event rate, though the ratio R_σ of scattering cross section of ν_μ and ν_τ to that of ν_e is roughly $R_\sigma = 1/6$ to $1/7$, because of the presence of the additional charged current process Fig. 6.4(b) in the case of the ν_e scattering. On the other hand, in the SNO charged current reaction, only ν_e can contribute to the process, as the charged current processes of other neutrinos require higher than available neutrino energies. Thus if the B neutrino flux predicted by the standard solar model is denoted by ϕ_{SSM}, the "effective" fluxes at these experiments, denoted by ϕ_{SK} and $\phi_{\mathrm{SNO}}(CC)$ should be given as

$$\phi_{\mathrm{SK}} = \phi_{\mathrm{SSM}} \times \{S + (1-S)R_\sigma\} \tag{6.160}$$

$$\phi_{\mathrm{SNO}}(CC) = \phi_{\mathrm{SSM}} \times S \tag{6.161}$$

where S is the survival probability (6.141). (For illustrative purpose, we have ignored the dependence of S and R_σ on the solar neutrino energy.) It is quite important to note that the observed value $\phi_{\mathrm{SK}} \simeq 2.4 \times 10^6$ $(/cm^2/s)$ is certainly larger than $\phi_{\mathrm{SNO}}(CC) \simeq 1.8 \times 10^6$ $(/cm^2/s)$. This means $S < 1$ and that among the solar neutrinos there surely exist the active neutrinos ν_μ and ν_τ, which strongly suggests the solar neutrino oscillation $\nu_e \rightarrow \nu_\mu, \nu_\tau$. (It also implies that the solar neutrino oscillation into a sterile state (states without weak interaction) is not favored.) Furthermore, equating (6.160) and (6.161) with the observed values, we get $\phi_{\mathrm{SSM}} = 5.4 \times 10^6$ $(/cm^2/s)$ and $S = 0.33$, for $R_\sigma = 1/6$. The obtained ϕ_{SSM} is quite consistent with the prediction of Bahcall et al., $(5 \pm 1) \times 10^6$ $(/cm^2/s)$.

The most recent remarkable experimental achievement is the confirmation of the solar neutrino oscillation in quite different circumstance, i.e. the terrestrial oscillation experiment of $\bar{\nu}_e$ produced by nuclear reactors in Japan, i.e. KamLAND. As we have seen in the subsection 6.5.1, the experimental setting is such that the oscillation is due to the smaller mass-squared difference Δm_{21}^2 and θ_{12}, just as in the solar neutrino oscillation, though the matter effect may be safely ignored, even though not negligible. Thus the formulae (6.129) and (6.130), similar to the one in the solar neutrino oscillation (6.141), gives a prediction on the deficit of $\bar{\nu}_e$ at the detector. The predicted deficit rate for the MSW-LMA solution, (6.158) and (6.159), is quite consistent with the observation, which started very recently (KamLAND collaboration, 2003).

Problems

6.1 Show that the relation (6.17) holds, to verify that a free Weyl fermion and a free Majorana fermion are equivalent, as far as they are massless.

6.2 Verify the relation (6.41) when (6.40) holds.

6.3 Show that the coefficient of the (irrelevant) operator (6.48) provides a Majorana mass for ν_L when the Higgs doublet ϕ develops its vacuum expectation value.

6.4 Verify the relation (6.66).

6.5 Verify that the adiabaticity condition (6.101) at the resonance point is equivalent to the condition (6.103).

6.6 State the intuitive physical meaning of (6.108).

Chapter 7

SUPERSYMMETRY

7.1 Supersymmetry and The Hierarchy Problem

Supersymmetry (SUSY for short) is a symmetry which connects particles and fields with different spin-statistics, i.e. bosons and fermions. When a Lagrangian density is invariant under a properly chosen transformation which connects particles whose spins differ by $1/2$, the theory is said to be supersymmetric. The supersymmetry is independent of any internal symmetry such as gauge symmetry, and therefore connects a pair of particles, "superpartners" of each other, with different spins but the same quantum numbers such as electric charge, weak isospin, color etc. As we will see later the generator of the supersymmetry transformation is known to commute with the generator of space-time translation P_μ (4-momentum), and therefore with P_μ^2. We thus learn that the superpartners have the same mass.

If we wish to apply the supersymmetry to the description of quarks and leptons, the supersymmetry clearly should be broken spontaneously or explicitly, as we have not seen any spin $= 0$ scalar particles with the same charge and the mass as those of quarks and leptons. Though it is not hard to break supersymmetry spontaneously, it easily happens that some of superpartners of light quarks or leptons are lighter than these particles, thus contradicting with the reality. In the Lagrangian, to put by hand the terms which break supersymmetry explicitly, such as the additional mass-squared term for the scalar electron, is also possible and an easy way. However, without any guiding principle, such arbitrarily introduced explicit SUSY breaking terms easily lead to an unacceptable prediction, such as too large flavor changing neutral currents. The detailed discussion on the mechanism of SUSY breaking is beyond the scope of this textbook, though we briefly discuss the mechanism of spontaneous SUSY breaking in a later section, and we just consider here how large the SUSY breaking mass scale can be. The scale will be fixed so that the SUSY works as a solution of the so-called hierarchy problem.

What is the hierarchy problem? Though it has been perfectly explaining all phenomena (except the recently reported neutrino oscillation), the standard model of elementary particles is widely believed not to be the final theory; in particular,

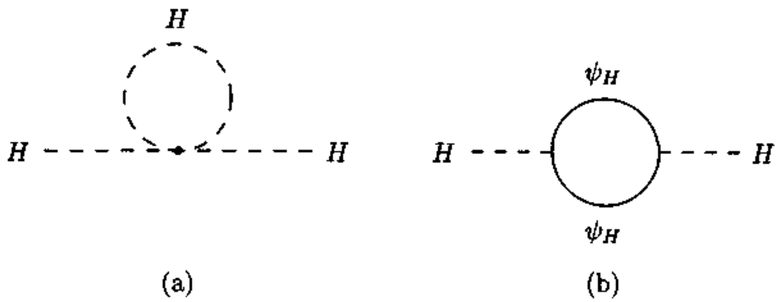


Fig. 7.1

it has too many free parameters and the gauge couplings are not really unified as the gauge group is not a simple group. We thus expect that the standard model is valid up to some energy (mass) scale Λ (a physical cutoff of the standard model), and at Λ the theory is replaced by more fundamental new theory incorporating the standard model, i.e. "beyond the standard model" or "new physics". The standard model is thus regarded as the "low" energy effective theory with the cutoff Λ. The cutoff Λ is M_{GUT} if the new physics is Grand Unified Theories (GUT), with e.g. $M_{GUT} \sim 10^{15}$GeV for $SU(5)$ GUT, and $M_{pl} \sim 10^{19}$GeV (Planck mass) if the new physics is a unified theory with gravity. These mass scales, M_{GUT}, M_{pl} are much higher than the weak scale M_W, the typical mass scale of the standard model. The hierarchy problem is the problem of how to maintain the hierarchy of these mass scales, i.e. $M_W \ll M_{GUT}, M_{pl}$. The problem is the most serious in the sector of scalar particle such as the Higgs particle, as the Higgs mass-squared get a huge quantum correction proportional to M_{GUT}^2 or M_{pl}^2. This large quantum correction manifest itself as a "quadratic divergence" Λ^2 if we calculate the quantum correction to the Higgs mass-squared in the effective theory, i.e. in the standard model: the problem of "quadratic divergence" (See Fig. 7.1(a)). Thus it is hard to maintain the hierarchy naturally. Actually, it is possible to adjust the bare mass of the Higgs, so that the renormalized Higgs mass remain to be the weak scale. It, however, will require a fine tuning of the bare parameter at the precision roughly of $(M_W/M_{GUT})^2 \sim 10^{-26}$, for instance, which we regard as unnatural.

Supersymmetry had been a subject of the theoretical research for rather long time, but its application to the particle physics was strongly motivated by the desire to solve the hierarchy problem. In fact, supersymmetric $SU(5)$ GUT was proposed by Sakai and Dimopoulos–Georgi (Sakai, 1981; Dimopoulos–Georgi, 1981) based on the motivation. Let us note that in the supersymmetric theory coupling constants of a particle and its superpartners are identical, and also that the Feynman rule provides an additional negative sign for the diagram with a loop of fermion. Thus the quadratic divergences from the two diagrams Fig. 7.1(a) and (b) are known to cancel with each other as long as the supersymmetry is exact. In this way, the hierarchy problem of the quadratic divergence is solved. As the matter of fact, the supersymmetry should be broken to the extent of "SUSY breaking mass scale" M_{SUSY}. We, however, are still free from the quadratic divergence, though we do have an logarithmic divergence, $\sim \alpha M_{SUSY}^2 \log\left(\frac{\Lambda}{M_W}\right)$. We regard the logarithmic correction as the order of αM_{SUSY}^2. Then the SUSY breaking scale should not be too high, since otherwise the hierarchy problem will arise again, i.e. $\alpha M_{SUSY}^2 \leq M_W^2$. This leads to a condition $M_{SUSY} \leq 1$ (TeV), roughly speaking.

It should be also stressed that another very important motivation to study the SUSY is that it almost automatically accommodates gravity. This is because the commutation relation of two infinitesimal SUSY transformations is just space-time translation. Thus, if SUSY is promoted to a local symmetry the resultant theory should be invariant under a local transformations of space-time coordinates, i.e. under the general coordinate transformations, thus leading to a supersymmetric gravity theory, supergravity.

Discussions in this chapter is, in many respects, based upon the classic textbook on the supersymmetry and supergravity by Wess and Bagger (Wess and Bagger, 1992).

7.2 Two-component Notation

Supersymmetry (SUSY) is in some sense a symmetry concerning (external) space-time coordinates, instead of internal symmetries; the fact that the commutation relation of SUSY transformations is space-time translation is the manifestation. Actually, as we will see later, SUSY transformation can be understood as a translation in a Grassmannian space, "superspace". Just as the Lorentz transformation, the SUSY transformation preserves (a little modified) chirality. Thus the most fundamental representation of SUSY has a definite chirality, and the usage of chiral (Weyl) fermion, i.e. the usage of the two-component notation is useful. We thus first summarize the notation and some useful relations in the two-component notation.

Weyl spinor, $\psi_{R,L} = \frac{1\pm\gamma_5}{2}\psi$, behave as irreducible representation of Lorentz group, since $[\Sigma_{\mu\nu}, \gamma_5] = 0$, with $\Sigma_{\mu\nu}$ being the generator of the Lorentz group. A Weyl spinor has a half degrees of freedom compared with a Dirac spinor and is expected to be described by a two-component spinor, in a suitable base of γ-matrices. To see this explicitly, we employ the Weyl base of γ-matrices, where γ_5 is a diagonal matrix, i.e. the same base as shown in (6.1)–(6.4);

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{7.1}$$

$$\sigma^\mu = (-I, \sigma_i), \tag{7.2}$$

$$\overline{\sigma}^\mu = (-I, -\sigma_i), \tag{7.3}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{7.4}$$

where I is a 2×2 unit matrix and σ_i $(i = 1,2,3)$ are Pauli matrices. Then the Weyl fermions are described by 2-component complex spinors as we expected:

$$\psi_R = \begin{pmatrix} \eta_\alpha \\ 0 \end{pmatrix}, \tag{7.5}$$

$$\psi_L = \begin{pmatrix} 0 \\ \overline{\xi}^{\dot\alpha} \end{pmatrix}, \quad (\alpha, \dot\alpha = 1, 2). \tag{7.6}$$

The Lorentz generators have the form of block diagonal

$$\Sigma^{\mu\nu} = \frac{i}{2}\begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \overline{\sigma}^{\mu\nu} \end{pmatrix} \quad (\mu \neq \nu), \tag{7.7}$$

$$\sigma^{\mu\nu} = \sigma^\mu\bar\sigma^\nu, \quad \overline{\sigma}^{\mu\nu} = \bar\sigma^\mu\sigma^\nu, \tag{7.8}$$

which clearly means that the chirality is preserved under the Lorentz transformation. It will be necessary to comment on the raising and lowering of the 2-component spinor indices α or $\dot\alpha$ and the meaning of the dot. We first note that the right-handed Weyl spinor, η_α, is transformed by a Lorentz transformation as $\eta'_\alpha = M_\alpha{}^\beta\eta_\beta$, where the 2×2 matrix

$$M = \exp(i\epsilon_{\mu\nu}\Sigma^{\mu\nu}), \tag{7.9}$$

with 6 transformation parameters $\epsilon_{\mu\nu}$ $(\epsilon_{\nu\mu} = -\epsilon_{\mu\nu})$. It should be noted that the M is an element of $SL(2,C)$. In fact, since the generators $\Sigma_{\mu\nu}$ are traceless $\det M = 1$ and also the 6 real degrees of freedom of $\epsilon_{\mu\nu}$ just coincide with the degree of $SL(2,C)$, $2\times 2^2 - 2$. We can easily show that in $SL(2,C)$ an invariant inner-product of two fundamental representations, i.e. doublets η_α and χ_α, is given by $\epsilon^{\alpha\beta}\eta_\alpha\chi_\beta$ ($\epsilon^{\alpha\beta}$: 2×2 Levi-Cività tensor, $\epsilon^{12} = -\epsilon^{21} = 1$), by use of the property $\det M = 1$. Thus the invariant product can be written as

$$\eta_\alpha\,\chi^\alpha, \tag{7.10}$$

once the raising of the lower index is defined as

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta. \tag{7.11}$$

In other words, the Levi-Cività tensor $\epsilon^{\alpha\beta}$ behaves as the metric tensor in the 2-component spinor space. Hereafter we will use the abbreviation, $\eta\chi \equiv \eta^\alpha\chi_\alpha = -\eta_\alpha\chi^\alpha = \chi^\alpha\eta_\alpha = \chi\eta$, $\overline{\eta\chi} \equiv \overline{\eta}_{\dot\alpha}\overline{\chi}^{\dot\alpha} = \overline{\chi\eta}$, for convenience.

Noting an identity $\overline{\sigma}^{\mu\nu} = -\epsilon\sigma^{\mu\nu *}\epsilon$, we find that the matrix of Lorentz transformation for a left-handed Weyl spinor should have the following indices,

$$-\epsilon^{\dot\alpha\dot\beta}\,(M^*)_{\dot\beta}{}^{\dot\gamma}\epsilon_{\dot\gamma\dot\delta}, \tag{7.12}$$

where the dot of the index denotes the complex conjugation of the quantity. To be consistent, the left-handed 2-component spinor should have the following upper dotted index

$$\psi_L = \begin{pmatrix} 0 \\ \overline{\xi}^{\dot{\alpha}} \end{pmatrix}. \tag{7.13}$$

We may also understand the origin of the dot as the result of charge-conjugation. Namely, the charge-conjugation interchanges the chirality of Weyl spinors, e.g. $\frac{1+\gamma_5}{2}(\psi_R)^C = 0$, while the charge conjugation always has the action of complex conjugation. Thus, when the right-handed spinor has an undotted index the left-handed spinor should have a dotted index. More explicitly, the charge-conjugation of a Dirac-spinor $\psi_D = \psi_R + \psi_L$ is given as,

$$\begin{aligned} (\psi_D)^C &= C\,\overline{\psi}_D^t = i\gamma^2\psi_D^* = i\gamma^2 \begin{pmatrix} \eta_\alpha \\ \overline{\xi}^{\dot{\alpha}} \end{pmatrix}^* \\ &= \begin{pmatrix} 0 & \epsilon_{\beta\alpha} \\ \epsilon^{\dot{\beta}\dot{\alpha}} & 0 \end{pmatrix} \begin{pmatrix} \overline{\eta}_{\dot{\alpha}} \\ \xi^\alpha \end{pmatrix} = \begin{pmatrix} \xi_\beta \\ \overline{\eta}^{\dot{\beta}} \end{pmatrix}, \end{aligned} \tag{7.14}$$

where $\overline{\eta}_{\dot{\alpha}} = (\eta_\alpha)^*$ etc. This clearly shows that when $\xi = \eta$ the spinor

$$\psi_M = \begin{pmatrix} \eta_\alpha \\ \overline{\eta}^{\dot{\alpha}} \end{pmatrix}, \tag{7.15}$$

is self-conjugate under the charge-conjugation,

$$(\psi_M)^C = \psi_M. \tag{7.16}$$

This specific "real" spinor is called Majorana spinor, and the particle described by the spinor is called Majorana particle.

Once the assignment of the spinor indices of Weyl spinors is fixed, the assignment of the indices of various 2×2 matrices are determined as,

$$(\sigma^\mu)_{\alpha\dot{\alpha}}, (\overline{\sigma}^\mu)^{\dot{\alpha}\alpha}, (\sigma^{\mu\nu})_\alpha{}^\beta, (\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}, \text{etc.} \tag{7.17}$$

The Lagrangians of renormalizable theories have bi-linear terms of spinors. Weyl spinors ψ_R and ψ_L behave as 2 and $\bar{2}$ representations of $SL(2,C)$, or equivalently $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of the Lorentz group, where $\frac{1}{2}$ and 0 are the "spins" of two $SU(2)$-like groups, contained in the Lorentz group. We then ask how bi-linear forms of Weyl fermions behave under the Lorentz group. The bi-linear form of two Weyl spinors with the same chirality decomposes into $(1,0)+(1,0)$ or $(0,1)+(1,0)$ representations, as is known from the recombination of the spin: $\frac{1}{2}\times\frac{1}{2} = 0+1$. More concretely, the singlet $(0,0)$ and the triplet $(1,0)$ representations formed by the product of two, e.g., right-handed spinors, η_1 and η_2, are written as

$$\epsilon^{\alpha\beta}\eta_{1\alpha}\eta_{2\beta} \equiv \eta_{1\alpha}\eta_2^\alpha = -\eta_1\eta_2, \quad (\sigma^{\mu\nu})^{\alpha\beta}\ \eta_{1\alpha}\eta_{2\beta}. \tag{7.18}$$

The reader can easily check that $(\sigma^{\mu\nu})^{\alpha\beta}$ is symmetric under the exchange of the spinor indices α and β, the property necessary for the triplet. This statement, however, is a little puzzling, as the number of possible combinations of μ and ν in the $\sigma^{\mu\nu}$ is 6, instead of 3. Actually, we find that not all of $\sigma^{\mu\nu}$ are independent; for instance σ^{01} and σ^{23} are essentially identical matrices. (In fact, if our space-time were Euclidean, this tensor satisfies the self-dual condition $\sigma^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}\sigma^{\kappa\lambda}$, while $\overline{\sigma}^{\mu\nu}$ satisfies the anti-self-dual condition.) Similarly, the product of two left-handed spinors decomposes into $(0,1)+(0,0)$ by the contractions with $(\overline{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}$ and $\epsilon_{\dot{\alpha}\dot{\beta}}$. In the four component-notation, where the right-handed spinors are written as $\psi_1 = (\eta_{1\alpha}, 0)^t$ and $\psi_2 = (\eta_{2\alpha}, 0)^t$, the singlet and the triplet discussed above correspond to the following bi-linear forms, $\psi_1^t C \psi_2$ and $\psi_1^t C \Sigma^{\mu\nu}\psi_2$. Then what corresponds to the Lorentz vector $\overline{\psi}_1\gamma^\mu\psi_2$? (Note that $\psi_1^t C\gamma_\mu\psi_2$ identically vanishes.) This time the bi-linear form is made by the mixture of the dotted and undotted indices, i.e. $-\overline{\eta}_{1\dot{\alpha}}\,(\overline{\sigma}^\mu)^{\dot{\alpha}\alpha}\eta_{2\alpha}$. Namely, $(\frac{1}{2},0)\times(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$ behaves as a Lorentz vector with 4 components.

We display in the Problem 7.1 some of formulae in the 2-component notation, which are frequently used and useful in the discussion below.

7.3 SUSY Algebra and SUSY Group

As already stated, supersymmetry (SUSY) is the symmetry under the exchange of bosons and fermions. Remarkably, it is not only a possible symmetry, but also is a unique symmetry, consistent with relativistic quantum field theory, besides internal symmetries such as gauge symmetry.

It had been known that generally the Lie algebra of possible symmetries of the S-matrix is restricted to the one composed of P_μ, $M_{\mu\nu}$, and Lorentz scalar generators which are the generators of

space-time translation,
Lorentz transformation,
gauge transformation,

respectively. Supersymmetry, therefore, seems to be not allowed as a symmetry. This no-go theorem (Coleman and Mandula, 1967), however, has been proved to be evaded, once the Lie algebra is extended so that it may contain anti-commutation relations as well as commutation relations, i.e. to an graded Lie algebra (Haag, Lopuszanski and Sohnius, 1975): the algebra now takes a form

$$Q, Q' = X, \quad [X, X'] = X'', \quad [Q, X] = Q'', ..., \tag{7.19}$$

where Q etc. and X etc. are, say, fermionic and bosonic generators. Since SUSY transformation connects fields whose spins differ by 1/2, the generators of SUSY naturally carry spin 1/2, and therefore satisfy the anti-commutation relation due to the Fermi-statistics, just as Q etc. do in (7.19).

Through such rigorous argument, it turns out that the SUSY algebra is given as

$$\{Q_\alpha,\ \overline{Q}_{\dot\alpha}\} = -2(\sigma^\mu)_{\alpha\dot\alpha}\ P_\mu, \tag{7.20}$$

$$\{Q_\alpha,\ Q_\beta\} = \{\overline{Q}_{\dot\alpha},\ \overline{Q}_{\dot\beta}\} = 0, \tag{7.21}$$

$$[P_\mu, Q_\alpha] = [P_\mu, \overline{Q}_{\dot\alpha}] = 0, \tag{7.22}$$

$$[P_\mu, P_\nu] = 0, \tag{7.23}$$

where Q_α and $\overline{Q}_{\dot\alpha}$ are complex conjugate, or hermitian conjugate in the sense of operators, of each other, and are Grassmannian operators, as is seen typically in (7.21). One may wonder why the anti-commutator in (7.21) should vanish, while that in (7.20) is non-vanishing. Just as the product of two doublets $\frac{1}{2}\times\frac{1}{2}$ decomposes into a triplet and a singlet $1 + 0$ in the calculus of spin, if $\{Q_\alpha,\ Q_\beta\}$ were non-vanishing, it generally should have contained two parts $C_{\{\alpha,\beta\}}$ and $C_{[\alpha,\beta]}$, i.e. the parts symmetric and anti-symmetric under the exchange of α and β, belonging to (1,0) and (0,0) representations of the Lorentz group, respectively. The anti-symmetric part, however, trivially vanishes, since $\{Q_\alpha,\ Q_\beta\}$ is symmetric under the exchange. Let us note that this situation changes in the case of extended SUSY with N ($N > 1$) kinds of SUSY transformations generated by Q^i_α (and $\overline{Q}_{\dot\alpha,i}$), where a "central charge" term of the form $\epsilon_{\alpha\beta}\ \hat{c}^{[i,j]}$, symmetric under the exchange $(\alpha, i) \leftrightarrow (\beta, j)$, is allowed to exist. In the case of simple SUSY $N = 1$, howver, the only possibility is the symmetric piece $C_{\{\alpha,\beta\}}$, proportional to $(\sigma^{\mu\nu})_{\alpha\beta}\ M_{\mu\nu}$. A rigorous argument leads to the conclusion that this term is not allowed to exist, being consistent with the SUSY algebra $[P_\mu, Q_\alpha] = 0$.

We thus know that the commutator of SUSY transformations provides a space-time translation, i.e.

$$[(\epsilon^\alpha Q_\alpha),\ (\overline{\epsilon}_{\dot\alpha}\overline{Q}^{\dot\alpha})] = [(\epsilon Q),\ (\overline{\epsilon}\overline{Q})] = -2(\epsilon^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\overline{\epsilon}^{\dot\alpha})\ P_\mu, \tag{7.24}$$

where ϵ^α, $\overline{\epsilon}^{\dot\alpha}$ are Grassmannian transformation parameters for SUSY and $\epsilon^\mu \equiv \epsilon^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\overline{\epsilon}^{\dot\alpha}$ denotes the amount of the infinitesimal space-time translation. Therefore, it is a rather trivial fact that once SUSY is promoted to a local symmetry the resultant theory inevitably has a local translational invariance, i.e. the invariance under the general coordinate transformations, thus leading to a SUSY theory including gravity, i.e. supergravity. In this book, we will restrict our argument to the global (rigid) supersymmetry where the transformation parameters are space-time independent.

Having the SUSY algebra, we may define a SUSY group, which is a set of transformations generated by the SUSY algebra, $e^{i(\epsilon^\mu p_\mu + \epsilon Q + \overline{\epsilon}\overline{Q})}$, where the generators

p_μ, Q and $\overline{Q}$ are the elements of SUSY algebra and $\epsilon^\mu, \epsilon, \bar{\epsilon}$ are the corresponding transformation parameters. We need representations of SUSY algebra, on which the transformations of SUSY group should act, so that we can assign our quarks and leptons and gauge bosons to some of the representations. In the $SU(2)$ gauge theory, for instance, the representations are concretely expressed as multiplets, such as doublet, triplet, etc., on which the group elements act in the form of matrices. Similarly, the representations of SUSY group can be expressed as "SUSY multiplets". As the SUSY transformation connects fields whose spins differ by 1/2, it is expected that each multiplet is made of "super-partners" of each other, such as a (spin 0) scalar and a (spin 1/2) fermion. SUSY generators Q and $\overline{Q}$ commute with p^2. Thus all members of a SUSY multiplet on their mass-shell should have a unique mass, and the SUSY multiplet can be classified by the mass M ($p^2 = M^2$). Let us first consider massless ($M = 0$) SUSY multiplets. The massless multiplets we discuss in this book are

$$\text{"chiral multiplet"}: \quad (A,\ \psi_{L,R}) \ \ (s = 0,\ s = 1/2) \tag{7.25}$$

$$\text{"vector multiplet"}: \quad (V_\mu,\ \lambda) \ \ (s = 1,\ s = 1/2), \tag{7.26}$$

where s denotes the spin. A is a complex scalar field and λ should be a Majorana fermion, corresponding to the real vector field V_μ. The chirality of the Weyl fermion of the chiral multiplet can be either L or R. We will be able to assign our quarks and leptons and Higgs to the component fields of the chiral multiplets, together with their super-partners. Gauge bosons, such as photon, are assigned to V_μ together with their superpartners, such as "photino". The important feature of the SUSY multiplets is that the (real) degrees of freedom of boson and fermion fields just coincide. This, however, seems to be not the case in the above multiplets; for instance, in the chiral multiplet $(A,\ \psi_{L,R})$, the complex scalar clearly has 2 degrees, while the Weyl fermion ψ_L seems to have 4 real degrees of freedom, at the first sight. This superficial puzzle may be resolved if we realize that the on-shell fermionic state has just the half degrees of freedom, due to the equation of motion. For instance a massless Weyl fermion $\psi_R = (\eta_\alpha,\ 0)^t$ with 4-momentum $p_\mu = (p, 0, 0, p)$ satisfies a Dirac (Weyl) equation in the momentum space

$$(\overline{\sigma}^\mu)^{\dot{\alpha}\alpha}\, p_\mu\, \eta_\alpha = 0 \quad \leftrightarrow \begin{pmatrix} -2p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{7.27}$$

which shows that the η_1 disappears and only η_2 remains. This argument, in turn, suggests that the SUSY algebra is closed only for on-shell states, or only by use of the equations of motion. In order to make the SUSY algebra close even for off-shell states, we therefore need to introduce additional bosonic states. Such additional states, i.e. 1 complex scalar field in the case of chiral multiplet and 1 real scalar field in the case of vector multiplet, should not have physical degrees of freedom, since they disappear once the equations of motion are applied. Thus these fields are called "auxiliary fields".

Concerning the SUSY multiplets of (on-shell) massive states with $M \neq 0$, the situation is rather different from the massless case. A general argument (see for instance (Wess and Bagger, 1992)) shows that the multiplets are generally "larger" than those in the massless case. Let us take the example of a massive SUSY multiplet with the highest spin 1. This multiplet has one $s = 1$, two $s = 1/2$ and one $s = 0$ component fields, though the SUSY transformation changes the spin of fields only by 1/2. This may not be so surprising if we recall the Higgs mechanism in ordinary gauge theories. For a gauge bosons to be massive, it needs to absorb a scalar field. Thus for a vector SUSY multiplet to be massive, it needs to absorb a chiral multiplet, which has inevitably a scalar component field. One remarkable feature of the SUSY theory is that, at the same time as the gauge boson becomes massive absorbing the scalar, gauge fermion (a superpartner of gauge boson, "gaugino") absorbs the chiral fermion of chiral SUSY multiplet to form a massive Dirac fermion (super-Higgs mechanism). For instance, suppose that the vector multiplet $(V_\mu,\ \lambda)$ is made of the photon and "photino", and the scalar component A of the chiral multiplet $(A,\ \psi_{L,R})$ is a charged scalar. Assuming that the real part of the A develops the vacuum expectation value (VEV) by a suitable scalar potential, the photon will absorb the imaginary part by the Higgs mechanism. Thus, we have a massive photon with 3 degrees of freedom and a massive scalar of 1 degree, i.e. the real part. As for the fermionic part, due to the super-Higgs mechanism, the photino λ and ψ form a massive Dirac particle, which should be the super-partner of the massive photon.

7.4 Superfield Formulation

We have formulated SUSY algebra and its representation, SUSY multiplets. The next task is to find the rule of SUSY transformation among the component fields of a SUSY multiplet, which is non-trivial. A powerful technique for such purpose has been invented, which makes the formulation of SUSY transformation and the construction of supersymmetric theories, i.e. the theories invariant under the SUSY transformation, almost automatic.

The formulation is based on the notion of "superfields", which are the fields defined on "superspace". Before going into the detail, let us recall that the generator of space-time translation, p_μ, acts on an ordinary field $\phi(x^\mu)$ as a differential operator $p_\mu \phi(x^\mu) = i\frac{\partial}{\partial x^\mu}\phi(x^\mu)$. Since $(\mathrm{SUSY})^2 \sim$ space-time translation, roughly speaking, and a Lorentz vector can be expressed as a bi-linear form of a spinor, $\epsilon^\mu = \epsilon^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}$ as suggested by (7.24), it will be natural to expect that the generators of SUSY transformations Q and $\overline{Q}$ act as a sort of translations in a "fermionic space", with anti-commuting Grassmannian coordinates θ_α, $\overline{\theta}_{\dot{\alpha}}$ ($\{\theta_\alpha, \theta_\beta\} = \{\overline{\theta}_{\dot{\alpha}}, \overline{\theta}_{\dot{\alpha}}\} = \{\overline{\theta}_{\dot{\alpha}}, \theta_\alpha\} = 0$). The space with orninary space-time coordinate and the Grassmannian coordinates, $(x^\mu,\ \theta_\alpha,\ \overline{\theta}_{\dot{\alpha}})$, is called superspace, and a field defined on the superspace is called superfield. Actually it turns out that the

SUSY generator is not a simple translation of θ and $\overline{\theta}$ coordinates, but contains a translation of x^μ, as well. Then a natural question is what is the differential operator of SUSY generator, corresponding to $p_\mu = i\frac{\partial}{\partial x^\mu}$?

Sometimes it is useful to regard a space (or space-time) as a "group manifold", which is a space, whose point can be identified with an element of the group. For instance, a point on the Earth, i.e. a 2-dimesional sphere S^2, may be obtained as the result of the action of 3-dimensional rotation group $SO(3)$ of the point on the north-pole. Hence the point can be identified with the element of $SO(3)$ group. This identification is not unique, as the rotation about the z-axis, $SO(2)$, does not move the north pole and can be freely added; S^2 may be regarded as a group manifold of $SO(3)/SO(2)$, whose degree is $3-1=2$, the same as that of S^2. More explicitly, a point $P(\theta,\phi)$ with polar coordinate (θ,ϕ) and Cartesian Coordinates $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ on a unit S^2 embedded in a 3-dimensional space is identified with an element of $SO(3)$, $G(\theta,\phi,\alpha) = R_3(-\phi)R_2(-\theta)R_3(\alpha)$ with (see also (4.179))

$$R_2(\theta_i) = \begin{pmatrix} c_i & 0 & -s_i \\ 0 & 1 & 0 \\ s_i & 0 & c_i \end{pmatrix}, \quad R_3(\theta_i) = \begin{pmatrix} c_i & s_i & 0 \\ -s_i & c_i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.28}$$

where $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$. In fact,

$$\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} = R_3(-\phi)R_2(-\theta)R_3(\alpha)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = R_3(-\phi)R_2(-\theta)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{7.29}$$

where we find that the $SO(2)$ rotation with angle α is irrelevant. Let $Q(\theta',\phi')$ be a point on the S^2, which is obtained by the action of a $SO(3)$ rotation $G(\kappa,\lambda,\beta)$ on the point $P(\theta,\phi)$. Then how can we express the transformation of space coordinates $(\theta,\phi) \to (\theta',\phi')$ by use of κ,λ,β? We note

$$\begin{aligned} \begin{pmatrix} \sin\theta'\cos\phi' \\ \sin\theta'\sin\phi' \\ \cos\theta' \end{pmatrix} &= G(\theta',\phi',\gamma)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = G(\kappa,\lambda,\beta)\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \\ &= G(\kappa,\lambda,\beta)\{G(\theta,\phi,\alpha)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\} = \{G(\kappa,\lambda,\beta)G(\theta,\phi,\alpha)\}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{7.30}$$

This means the transformation of the space coordinates is completely determined by the multiplication rule of group elements: $G(\theta',\phi',\gamma)$
$= G(\kappa,\lambda,\beta)G(\theta,\phi,\alpha)$. Such determined transformation automatically satisfies all properties of $SO(3)$ group, by construction.

Similarly, we can establish the transformation of superspace coordinates $(x,\theta,\bar{\theta})$. Let $G(x^\mu,\theta,\overline{\theta}) = e^{i(x^\mu p_\mu + \theta Q + \overline{\theta Q})}$ be an element of SUSY group and identify it with a coordinate $(x^\mu, \theta, \overline{\theta})$. By use of the SUSY algebra (7.20)–(7.23) and the well-known relation $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$ we get (since higher commutators do not contribute)

$$G(0, \epsilon, \overline{\epsilon}) \cdot G(x^\mu, \theta, \overline{\theta}) = G(x^\mu + i\theta\sigma^\mu\overline{\epsilon} - i\epsilon\sigma^\mu\overline{\theta}, \theta + \epsilon, \overline{\theta} + \overline{\epsilon}). \tag{7.31}$$

¿From the identification we readily get a transformation property of the superspace coordinates due to the action of $G(0, \epsilon, \overline{\epsilon})$,

$$(x^\mu, \theta, \overline{\theta}) \quad \rightarrow \quad (x^\mu + i\theta\sigma^\mu\overline{\epsilon} - i\epsilon\sigma^\mu\overline{\theta}, \theta + \epsilon, \overline{\theta} + \overline{\epsilon}), \tag{7.32}$$

where $\theta\sigma^\mu\overline{\epsilon} = \theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\overline{\epsilon}^{\dot{\alpha}}$, etc. Thus we can conclude that the action of the SUSY transformation $G(0, \epsilon, \overline{\epsilon})$ on a superfield is equivalent to the application of $e^{i(\epsilon Q + \overline{\epsilon}\overline{Q})}$ with the following differential operators, corresponding to the (i times) SUSY generators,

$$iQ_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu)_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_\mu, \tag{7.33}$$

$$i\overline{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu. \tag{7.34}$$

The nice thing to use the group-manifold approach is that such constructed differential operators automatically satisfies the SUSY algebra, just because it is based on the group element itself. For instance if $G_1 G_2 = G_3$ holds among the elements of SUSY group, multiplying $G(x^\mu, \theta, \overline{\theta})$ from the right of both sides and identifying $G(x^\mu, \theta, \overline{\theta})$ as a superfield ϕ and G_1, G_2, G_3 with the corresponding group elements with differential operators, we immediately get $G_1 G_2 \phi = G_3 \phi$. In fact, we can directly confirm that the differential operators (7.33) and (7.34) satisfy

$$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = -2i(\sigma^\mu)_{\alpha\dot{\alpha}}\ \partial_\mu, \quad \{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0, \tag{7.35}$$

to be consistent with (7.20) and (7.21).

We will now see how the component fields of SUSY multiplets are contained in the superfields. Superfields containing the irreducible representation of SUSY algebra, i.e. (massless) chiral multiplet and the (massless) vector multiplet, are called chiral superfield and vector superfield. Before going to the discussion of each superfield, we first consider the most general complex superfield, $\phi(x, \theta, \overline{\theta})$ (the Lorentz and spinor indices of the coordinates have been suppressed). The important feature of the superfied is that, when it is Taylor-expanded with respect to Grassmannian coordinates θ, $\overline{\theta}$, it contains only a finite number of terms, whose coefficients are functions solely of x, i.e. ordinary fields. This termination of the expansion is due to the anti-commuting property of the Grassmannian coordinates; e.g. $\theta_\alpha\theta_\beta\theta_\gamma = 0$ as the spinor indices α, β, γ take only 1 or 2. Thus the general superfield can be expanded as

$$\phi(x, \theta, \overline{\theta}) \quad = \quad C(x) + \theta\chi(x) + \overline{\theta}\overline{\chi'}(x) + \theta\theta M(x) + \overline{\theta\theta}N(x)$$

$$+ \quad \theta\sigma^\mu\overline{\theta}V_\mu(x) + \theta\theta\overline{\theta\lambda}(x) + \overline{\theta\theta}\theta\psi(x) + \theta\theta\overline{\theta\theta}D(x), \tag{7.36}$$

where $\theta\theta = \theta^\alpha\theta_\alpha$ etc.. The bosonic component fields, C, M, N, V_μ, D, have in total $\binom{4}{0} + \binom{4}{2} + \binom{4}{4} = 8$ complex degrees of freedom, while the fermionic component fields, $\chi, \overline{\chi'}, \overline{\lambda}, \psi$, have $\binom{4}{1} + \binom{4}{3} = 8$ degrees of freedom. Thus we have checked the coincidence of bosonic and fermionic degrees of freedom, as expected from SUSY. Altogether we have $8 + 8 = 16$ degrees of freedom. This may be easily understood, since each term of the Taylor expansion can be written, in general, in a form $c_{ijkl}(x)\ (\theta_1)^i(\theta_2)^j(\overline{\theta}_{\dot{1}})^k(\overline{\theta}_{\dot{2}})^l$. We have $2^4 = 16$ choices of the coefficient functions $c_{ijkl}(x)$, as i, j, k, l take either 0 or 1.

The deviation of the superfield $i\delta\phi(x) = \phi'(x) - \phi(x)$, under an infinitesimal transformation, $\phi'(x) = \exp(i(\epsilon Q + \overline{\epsilon Q}))\ \phi(x)$, with infinitesimal Grassmannian parameters ϵ and $\overline{\epsilon}$ can be easily calculated by use of the differential operators (7.33) and (7.34) as

$$\delta\phi = i(\epsilon Q + \overline{\epsilon Q})\phi = \{\epsilon^\alpha(\frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu)_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_\mu) - \overline{\epsilon}^{\dot{\alpha}}(-\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu)\}\phi(x). \tag{7.37}$$

Writing $\delta\phi(x) = \delta C(x) + \theta\delta\chi(x) + \cdots$, the infinitesimal deviation of some of component fields can be read off as

$$\begin{aligned}
\delta C &= \epsilon\chi + \overline{\epsilon}\,\overline{\chi}, && (7.38)\\
\delta\chi_\alpha &= 2\epsilon_\alpha M + (\sigma^\mu)_{\alpha\dot{\alpha}}\overline{\epsilon}^{\dot{\alpha}}(V_\mu + i\partial_\mu C), && (7.39)\\
\delta\overline{\chi'}^{\dot{\alpha}} &= 2\overline{\epsilon}^{\dot{\alpha}}N + \epsilon^\alpha(\sigma^\mu)_\alpha{}^{\dot{\alpha}}(V_\mu - i\partial_\mu C), && (7.40)\\
\delta D &= \frac{i}{2}(\sigma^\mu)_{\alpha\dot{\alpha}}(\epsilon^\alpha\partial_\mu\overline{\lambda}^{\dot{\alpha}} - (\partial_\mu\psi^\alpha)\overline{\epsilon}^{\dot{\alpha}}). && (7.41)
\end{aligned}$$

We learn from the above exercise the following things, which play quite important roles when we attempt to construct a supersymmetric theory, i.e. a theory invariant under the SUSY transformation.

(a) The dimensionality of the component fields

As is suggested by (7.31), the bi-linear form of the Grassmannian coordinates θ_α and $\overline{\theta}_{\dot{\alpha}}$ behaves as the ordinary space-time coordinate x_μ. Hence, the θ_α and $\overline{\theta}_{\dot{\alpha}}$ should have mass dimension $d = -1/2$. The generators of SUSY transformation Q_α and $\overline{Q}_{\dot{\alpha}}$ therefore have mass dimension $d = 1/2$. This means by the SUSY transformation a component field with mass dimension d is transformed into a field with mass dimension $d + \frac{1}{2}$ or the space-time derivative of a field of lower mass dimension $d - \frac{1}{2}$. We also learn that the component field accompanied by a higher power of θ_α and $\overline{\theta}_{\dot{\alpha}}$ has higher mass dimension.

(b) The property of D-term

The above argument suggests that the "highest component" $D(x)$, or "D-term", proportional to the highest power $\theta\theta\overline{\theta\theta}$, has the highest mass dimension. For instance, if we assign the Lorentz vector field V_μ the mass dimension $d = 1$, the D field has $d = 2$. As the quadratic term of $D(x)$ already has $d = 4$, we cannot include any space-time derivative in the free Lagrangian of the D field, thus making the D an auxiliary field, i.e. a field without dynamical degree of freedom and is describable in terms of other physical fields, once the equation of motion for the field is used.

As the D field has the highest mass dimension, the infinitesimal deviation of the $D(x)$ under the SUSY transformation, is inevitably the space-time derivative of the field of lower mass dimension. In fact δD in (7.41) can be rewritten as a total derivative, noting that ϵ and $\bar{\epsilon}$ are space-time independent constant: $\delta D = \partial_\mu\{\frac{i}{2}\epsilon^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\overline{\lambda}^{\dot{\alpha}} - \frac{i}{2}\psi^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\}$. This means that if the Lagrangian of a theory can be regarded as the D-trem of some superfield, the action is supersymmetric, since a total derivative term of the Lagrangian is irrelevant. (If space-time has some compact dimension, as usually assumed in higher dimensional theories, the derivative term may not be ignored for non-trivial boundary conditions of fields.) It is suggestive to note that a similar thing happens when a theory is invariant under a space-time translation, i.e. also in this case the Lagrangian itself is not invariant under the translation, but the deviation can be written as a total derivative, thus making the action invariant.

(c) The property of closure

To find out the multiplication rule of SUSY multiplets, namely a rule to reconstruct a new SUSY multiplet out of the product of two SUSY multiplets, is a non-trivial task. In the language of the superfield, however, the multiplication rule is quite simple, i.e. we can just multiply two superfields. If ϕ_1 and ϕ_2 are two superfields, their product $\phi_3(x,\theta,\overline{\theta}) = \phi_1(x,\theta,\overline{\theta})\cdot\phi_2(x,\theta,\overline{\theta})$ can be Taylor expanded again, and therefore behaves as a new superfield ("closure" property of superfields under the multiplication). Each component field of ϕ_3 is written as a bi-linear term of the component fields of ϕ_1 and ϕ_2, which provides us the multiplication rule. This is essentially due to the fact that the SUSY generator is a linear differential operator and the chain rule $i(\epsilon Q+\overline{\epsilon Q})\phi_3 = \{i(\epsilon Q+\overline{\epsilon Q})\phi_1\}\cdot\phi_2+\phi_1\cdot\{i(\epsilon Q+\overline{\epsilon Q})\phi_2\}$ holds.

By use of these properties, we can easily construct (some part of) the action of a supersymmetric theory,

$$S = \int d^4x\ f(\phi_1,\phi_2,\cdots)|_D, \tag{7.42}$$

where $f(\phi_1,\phi_2,\cdots)|_D$ denotes the D-term (the highest component) of the polynomial function f of superfields, $\phi_1,\phi_2,\cdots$, which is again a superfield. The infinitesimal deviation of the D-term is a total derivative and the action is SUSY invariant. Since SUSY transformation is a sort of translation in the superspace, it is naturally expected that the integral of the Lagrangian over the entire superspace yields a SUSY invariant action. In fact, we find that the action can be equivalently written as

$$S = \int d^4x \, d^2\theta \, d^2\overline{\theta} \; f(\phi_1, \phi_2, \cdots). \tag{7.43}$$

Let us note that the integration over Grassmannian variables is equivalent to the differentiation in terms of these variables. Thus $d^2\theta \; d^2\overline{\theta}$ is equivalent to extracting the D-term.

7.5 Chiral superfield

The general superfield turns out to be reducible representation of SUSY, as it contains fields with spin $s = 0, 1/2, 1$ together, while SUSY transformation changes the spin only by $1/2$. It is thus important to identify the irreducible representations of SUSY. There are two kinds of irreducible representation, i.e. chiral superfield and vector superfield, which contain, as their component fields the chiral multiplet and the vector multiplet, respectively.

In this section, we will discuss the chiral superfield, which contain component fields with $s = 0$ and $1/2$. Naive guess is that we can get it by adopting a "chiral" superspace, whose Grassmannian coordinate is restricted to a Weyl fermion with definite chirality, θ or $\overline{\theta}$, but not the both, e.g. space with coordinates (x_μ, θ_α). Then the power series expansion with respect to θ_α will terminate at the quadratic term $\theta\theta$, and the appearance of the $s = 1$ vector field V_μ will be evaded. Thus we are tempted to impose a constraint in order to define, e.g., a "right-handed" chiral superfield ϕ, $\frac{\partial}{\partial \overline{\theta}^{\dot{\alpha}}} \; \phi = 0$, so that ϕ does not depend on $\overline{\theta}$. This constraint equation, however, does not work, since the condition is not compatible with SUSY, i.e. $\{Q_\alpha, \frac{\partial}{\partial \overline{\theta}^{\dot{\alpha}}}\} \neq 0$. We then have to find out a suitable differential operator, which (anti-)commutes with the SUSY generators. The desirable operators turn out to be

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i(\sigma^\mu)_{\alpha\dot{\alpha}} \, \overline{\theta}^{\dot{\alpha}} \, \partial_\mu, \tag{7.44}$$

$$\overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \overline{\theta}^{\dot{\alpha}}} - i\theta^\alpha \, (\sigma^\mu)_{\alpha\dot{\alpha}} \, \partial_\mu. \tag{7.45}$$

It is straightforward to check that these operators anti-commute with SUSY generator,

$$\{Q_\alpha, D_\beta\} = \{\overline{Q}_{\dot{\alpha}}, D_\alpha\} = \{Q_\alpha, \overline{D}_{\dot{\alpha}}\} = \{\overline{Q}_{\dot{\alpha}}, \overline{D}_{\dot{\beta}}\} = 0. \tag{7.46}$$

Let us note that these "SUSY-covariant" derivatives D_α and $\bar{D}_{\dot{\alpha}}$ are quite similar to iQ_α and $i\bar{Q}_{\dot{\alpha}}$ in (7.33) and (7.34), except the sign in front of the space-time derivative. This is not an accident. Let us recall that the differential operator Q and $\overline{Q}$ were read off from the relation, $G(0, \epsilon, \overline{\epsilon}) \cdot G(x^\mu, \theta, \overline{\theta}) = G(x^\mu + i\theta\sigma^\mu\overline{\epsilon} - i\epsilon\sigma^\mu\overline{\theta}, \theta + \epsilon, \overline{\theta} + \overline{\epsilon})$, where we identified the "left multiplication" of $G(0, \epsilon, \overline{\epsilon})$ as an action of the element of SUSY group. What happens if the multiplication of

$G(0, \epsilon, \bar{\epsilon})$ is from the right? The only change will be that the sign of the space-time translation, proportional to ϵ or $\bar{\epsilon}$, is just opposite to the case of left-multiplication, i.e. $-i\theta\sigma^\mu\bar{\epsilon} + i\epsilon\sigma^\mu\bar{\theta}$. Thus we find that the operators D and $\overline{D}$ can be regarded as the differential operators corresponding to the right-multiplication. It is now trivial that they commute with the SUSY generators, just because the right- and left-multiplications are mutually independent.

Now, for instance, a right-handed chiral superfield ϕ can be defined as

$$\overline{D}_{\dot{\alpha}}\ \phi = 0. \tag{7.47}$$

(The left-handed superfield is defined similarly by replacing $\bar{D}_{\dot{\alpha}}$ by D_α.) As expected, $\overline{D}_{\dot{\alpha}}\theta^\alpha = 0$ means that the superfield ϕ may depend on the right-handed coordinate θ^α. The space-time coordinate x^μ, however, is not a suitable coordinate to describe ϕ, as $\overline{D}_{\dot{\alpha}}x^\mu = -i\ \theta^\alpha\ (\sigma^\mu)_{\alpha\dot{\alpha}}$ does not vanish. We note that the bi-linear form of θ and $\bar{\theta}$ behaving as a Lorentz vector $\theta\sigma^\mu\bar{\theta}$ satisfies $\overline{D}_{\dot{\alpha}}(\theta\sigma^\mu\bar{\theta}) = \theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}$. We thus find the suitable Lorentz vector to describe ϕ is $y^\mu \equiv x^\mu + i\ (\theta\sigma^\mu\bar{\theta})$: $\overline{D}_{\dot{\alpha}}\ y^\mu = 0$. In this way, the chiral superfield should be a function of coordinates $(y^\mu,\ \theta_\alpha)$, and can be expanded in a power series of θ as

$$\begin{aligned}
\phi(y,\ \theta) &= A(y) + \sqrt{2}\ \theta\psi(y) + \theta\theta F(y) \\
&= e^{i(\theta\sigma^\mu\bar{\theta})\partial_\mu}\ (A(x) + \sqrt{2}\ \theta\psi(x) + \theta\theta F(x)) \\
&= A(x) + i\ (\theta\sigma^\mu\bar{\theta})\ \partial_\mu A(x) - \frac{1}{4}\ \theta\theta\bar{\theta}\bar{\theta}\ \Box A(x) + \sqrt{2}\ \theta\psi(x) \\
&- \frac{i}{\sqrt{2}}\theta\theta\ (\partial_\mu\psi)\sigma^\mu\bar{\theta} + \theta\theta F(x),
\end{aligned} \tag{7.48}$$

where $\Box \equiv \partial^\mu\partial_\mu$ and we have used the property that the power series expansion of $e^{i(\theta\sigma^\mu\bar{\theta})\partial_\mu}$ ends up at $\{(\theta\sigma^\mu\bar{\theta})\partial_\mu\}^2$, because of the anti-commuting property of θ and $\bar{\theta}$. In the derivation, some relations listed in Problem 7.1 may be useful. The component field F has appeared in addition to the expected scalar and fermion fields A and ψ. The field F has mass dimension $d = 2$, and is anticipated to be an auxiliary field, just as the D field in a general superfield. Let us note that in the third line of the right hand side of (7.48), derivative terms have appeared though there is no derivative of the chiral superfield at the beginning. This is due to the "non-locality" of the coordinate y^μ, which has a deviation $i(\theta\sigma^\mu\bar{\theta})$ from x^μ, and will play an important role in the construction of the kinetic term for the component fields.

In terms of the coordinates y^μ, θ^α, the infinitesimal SUSY transformation reads as

$$i\{(\epsilon Q + \bar{\epsilon}\overline{Q})\}\ \phi(y,\theta) = (\epsilon^\alpha\frac{\partial}{\partial\theta^\alpha} + 2i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}\partial_\mu)\ \phi(y,\theta), \tag{7.49}$$

where $\partial_\mu = \partial/\partial y^\mu$. It is now easy to derive the transformation property of each component field:

$$
\begin{aligned}
\delta A &= \sqrt{2}\epsilon\psi, && (7.50)\\
\delta\psi &= \sqrt{2}i\sigma^\mu\bar{\epsilon}\partial_\mu A + \sqrt{2}\epsilon F, && (7.51)\\
\delta F &= -\sqrt{2}i\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi. && (7.52)
\end{aligned}
$$

We realize that again the deviation of the "highest" component field F can be written as a total derivative (as $\bar{\epsilon}$ is y^μ-independent), which becomes important in the construction of supersymmetric theory of chiral superfields.

In order to see the multiplication rule for chiral superfields, it is suggestive to recall that multiplication of holomorphic functions $f(z)$ and $g(z)$ gives another holomorphic function $h(z) = f(z)g(z)$, while the multiplication of holomorphic and anti-holomorphic functions is no longer holomorphic nor anti-holomorphic, but is a general complex function. A complete analogy holds in the multiplication rule for chiral and anti-chiral superfields, i.e.,

$$
\begin{aligned}
\text{chiral} \times \text{chiral} &= \text{chiral},\\
\text{anti-chiral} \times \text{anti-chiral} &= \text{anti-chiral}, \qquad (7.53)\\
\text{chiral} \times \text{anti-chiral} &= \text{general}.
\end{aligned}
$$

Thanks to the multiplication rule, especially the closure property among chiral or anti-chiral superfields, it is quite easy to form a SUSY invariant monomials of such superfields,

$$
\int d^4y\{\phi_1(y,\theta)\phi_2(y,\theta)\cdots\phi_n(y,\theta)\}|_F \sim \int d^4y d^2\theta\{\phi_1\phi_2\cdots\phi_n\}, \qquad (7.54)
$$

where F denotes the extraction of the F-term of the chiral superfield obtained by the multiplication of the chiral superfields $\phi_1(y,\theta)$ etc., which is a total derivative with respect to y^μ. This procedure will be used in the formation of the "superpotential" which is discussed below. In (7.54), we may finally replace y^μ by the ordinary space-time coordinate x^μ.

7.6 Wess–Zumino Model

So called ϕ^4 theory is often discussed as a prototype theory of quantum field theory for the purpose of getting some essential ingredients of the quantum field theory. In this section, we discuss the supersymmetric extension of the ϕ^4 theory, a supersymmetric model of chiral multiplet $(A_i,\ \psi_i)$ $(i = 1, 2, \cdots)$, called Wess–Zumino model. As the new feature of the Wess–Zumino model, we have not only self-interaction of scalar fields but also Yukawa-type couplings between scalars and fermions. As the consequence of the SUSY, the coupling constants of such interactions are identical, and there should be mass degeneracy between the scalars and the fermions, unless there is SUSY breaking.

• **The kinetic term**

To get kinetic terms for scalars and fermions, it is clear that we need quadratic terms of chiral superfields. We learn from ordinary field theories that kinetic terms are those which are hermitian by themselves. We thus expect the multiplication of a chiral superfield and its hermitian conjugation, i.e. a anti-chiral superfield, provides the desirable kinetic term. This turns out to be the case, since the difference between y^μ and $y^{\mu\dagger}$ yields the non-locality, and therefore the kinetic term. (As already has been pointed out and we will see below more concretely, a simple square of a chiral superfield just provides a quadratic term in "superpotential", but not a kinetic term.) In fact, the multiplication yields for a given chiral superfield ϕ

$$\begin{aligned} \phi(y,\theta)^\dagger\phi(y,\theta) &= (e^{i(\theta\sigma^\mu\bar{\theta})\partial_\mu}\ \phi(x,\theta))^\dagger\ (e^{i(\theta\sigma^\mu\bar{\theta})\partial_\mu}\ \phi(x,\theta)) \\ &= A(x)^*A(x) + \sqrt{2}\theta\psi(x)A(x)^* + \sqrt{2}\overline{\theta\psi}(x)A(x) + \cdots \\ &+ \theta\theta\overline{\theta\theta}[-\frac{1}{4}A^*\Box A - \frac{1}{4}(\Box A^*)A + \frac{1}{2}(\partial_\mu A^*)(\partial^\mu A) \\ &+ \frac{i}{2}(\partial_\mu\overline{\psi})\overline{\sigma}^\mu\psi - \frac{i}{2}\overline{\psi}\overline{\sigma}^\mu(\partial_\mu\psi) + F^*F]. \end{aligned} \tag{7.55}$$

According to the multiplication rule, this product behaves as a general superfield. Thus taking the D-term we readily obtain the desirable SUSY invariant kinetic term for the set of chiral superfields:

$$\begin{aligned} \mathcal{L}_{kin} &= \phi_i^\dagger\phi_i|_D \\ &= (\partial_\mu A_i^*)(\partial^\mu A_i) - i\overline{\psi}_i\overline{\sigma}^\mu\partial_\mu\psi_i + F_i^*F_i + (\text{total derivative}). \end{aligned} \tag{7.56}$$

As we expected from dimensional counting, the field F has no derivative in this "kinetic" term and therefore is an auxiliary field. Namely, F does not propagate by itself, unless it couples with propagating fields through interactions. The remaining physical fields, A and ψ have just ordinary forms of kinetic terms.

• **The superpotential**

The aforementioned kinetic term is a free Lagrangian and does not contain any interactions. Let us now consider the self interaction of chiral superfields. Just as the self-interaction of scalar fields A_i are described by scalar potential $V(A)$, supersymmetric "self-"interaction is generally describable in terms of a polynomial of chiral superfield $W(\phi)$, called superpotential:

$$W(\phi) = \lambda_i\phi_i + \frac{1}{2}m_{i,j}\phi_i\phi_j + \frac{1}{3}g_{i,j,k}\phi_i\phi_j\phi_k. \tag{7.57}$$

¿From the closure property under the multiplication, the superpotential itself should be a chiral superfield. Thus $\int d^4x\ W(\phi)|_F$ (after the replacement of $y^\mu \to x^\mu$) should be SUSY invariant. Let the mass dimension of $W(\phi)$ be d_W. Then its F-term should have a mass dimension $d+1$, as $\theta\theta$ carries $d=1$. This is why the superpotential ends up at cubic terms of chiral superfields. Namely if we introduce terms higher than cubic, such as quartic terms, the mass dimension of the F-term should exceed 4, thus leading to a non-renormalizable theory. We will see below, more concretely, that such cubic superpotential eventually yields quartic self-interaction of scalar fields and Yukawa-type couplings.

Let us now calculate the F-term of the superpotential W. There is a systematic way to derive the F-term. We first calculate the F-term of a monomial $(\phi_1\phi_2\cdots\phi_n)|_F$. The F component is the sum of the terms, which are quadratic in the Grassmannian coordinates θ. To get the quadratic terms there are only two possibilities. One possibility is to pick up F_i component from some chiral superfield ϕ_i and take scalar components from all remaining chiral superfields:

$$A_1\cdots A_{n-1}F_n + A_1\cdots A_{n-2}F_{n-1}A_n + \cdots = \frac{\partial(A_1A_2\cdots A_n)}{\partial A_i}\,F_i. \tag{7.58}$$

The right hand side means that such term can be compactly expressed in terms of the first derivative of the monomial, obtained by replacing all chiral superfields by their scalar components. The next possibility is to take fermions fron two different places of the monomial and take scalar fields from all remaining chiral superfields:

$$-A_1\cdots\psi^\alpha_{n-1}\psi_{n\alpha} - A_1\cdots\psi^\alpha_{n-2}A_{n-1}\psi_{n\alpha} - \cdots = -\frac{1}{2}\frac{\partial^2(A_1A_2\cdots A_n)}{\partial A_i\partial A_j}\,\psi_i\,\psi_j. \tag{7.59}$$

As this compact way of writing is easily known to be valid for arbitrary monomials and because differential operators are linear operators, this procedure is applicable for an arbitrary superpotential. Hence, the F-term of the superpotential can be written as

$$W(\phi)|_F = \frac{\partial W(A)}{\partial A_i}\,F_i - \frac{1}{2}\frac{\partial^2 W(A)}{\partial A_i\partial A_j}\,\psi_i\,\psi_j. \tag{7.60}$$

The whole Lagrangian now reads

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{kin} - \{W(\phi)|_F + h.c.\} \\
&= (\partial_\mu A_i^*)(\partial^\mu A_i) - i\overline{\psi}_i\overline{\sigma}^\mu\partial_\mu\psi_i + F_i^*F_i \\
&- \{\frac{\partial W(A)}{\partial A_i}\,F_i - \frac{1}{2}\frac{\partial^2 W(A)}{\partial A_i\partial A_j}\,\psi_i\psi_j + h.c.\}.
\end{aligned} \tag{7.61}$$

The auxiliary field does not possess an independent dynamical freedom by itself, and is expected to be eliminated from the Lagrangian, if we wish. In fact, by use of Euler–Lagrange equation for F_i, F_i can be expressed in terms of scalar fields, as there is no derivative term of the auxiliary field in the Lagrangian. For such purpose we first write down the terms containing F_i fields in $\mathcal{L}$,

$$F_i^*F_i - (\frac{\partial W(A)}{\partial A_i}\,F_i + h.c.) = (F_i^* - \frac{\partial W(A)}{\partial A_i})(F_i - (\frac{\partial W(A)}{\partial A_i})^*) - |\frac{\partial W(A)}{\partial A_i}|^2. \tag{7.62}$$

By use of the Euler–Lagrange equation, the quantities inside the bracket just Disappear:

$$F_i^* = \frac{\partial W(A)}{\partial A_i} \tag{7.63}$$

Hence, the (ordinary) scalar potential V(A) is given as

$$V(A) = |\frac{\partial W(A)}{\partial A_i}|^2. \tag{7.64}$$

In this way the Lagrangian of the Wess–Zumino model can be rewritten in terms of only physical fields as

$$\begin{aligned}\mathcal{L} &= (\partial_\mu A_i^*)(\partial^\mu A_i) - i\overline{\psi}_i\overline{\sigma}^\mu\partial_\mu\psi_i \\ &+ \frac{1}{2}(\frac{\partial^2 W(A)}{\partial A_i \partial A_j}\ \psi_i\psi_j + h.c.) \\ &- |\frac{\partial W(A)}{\partial A_i}|^2. \end{aligned} \tag{7.65}$$

The first line at the right hand side is ordinary kinetic term and the second and the third lines denote Yukawa coupling and scalar potential, respectively.

- **Solving the hierarchy problem**

To see how SUSY works as a mechanism to solve the hierarchy problem or the problem of quadratic divergence, we will consider the Wess–Zumino model of a single chiral multiplet, with superpotential $W(\phi) = \frac{1}{2}m\ \phi^2 - \frac{1}{3}\ g\phi^3$. The Lagrangian is explicitly written as

$$\begin{aligned}\mathcal{L} &= (\partial_\mu A^*)(\partial^\mu A) - i\overline{\psi}\overline{\sigma}^\mu\partial_\mu\psi \\ &+ [(\frac{1}{2}m - gA)\ \psi^2 + h.c.] \\ &- |mA - gA^2|^2. \end{aligned} \tag{7.66}$$

Thus we know that the complex scalar A has a mass m and the fermion ψ has the same Majorana mass m. Thus the degeneracy of the masses of boson and fermion is realized, as an important consequence of SUSY. We further note that the Yukawa coupling $A\psi^2$ and the scalar self-interaction $(A^*A)^2$ have identical coupling constant g. This is another important consequence of SUSY. One remark here is that the scalar self-coupling is proportional to g^2, not g as in the Yukawa coupling, and SUSY seems to be not manifest. One may convince oneself that SUSY is really maintained, by noting that the self-interaction is originally due to the coupling between scalar A and auxiliary field F, $2gFA^2 + h.c.$, whose coupling constant is proportional to g, thus preserving SUSY (see Fig. 7.2).

As was mentioned in the beginning of this Chapter, SUSY has a remarkable feature to stabilize the scalar mass under the quantum correction, i.e. the problem of

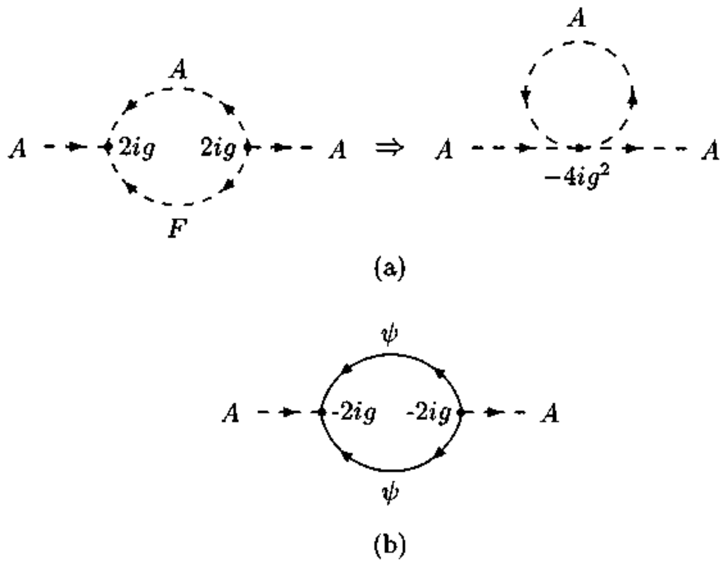


Fig. 7.2

quadratic divergence is absent in SUSY theories. Let us now explicitly show that the cancellation of the quadratic divergence does happen in the Wess–Zumino model. What we calculate is the Feynman diagrams (a) and (b) shown in Fig. 7.2. In order to calculate the diagram with ψ-exchange, we need a propagator for the fermion ψ. Since we are familiar with the propagator for 4-componet spinor, and as ψ has a Majorana mass term $\frac{1}{2}m\,\psi^2 + h.c.$, let us define a Majorana fermion $\psi_M = \psi + (\psi)^C$, with ψ being understood as a 4-vector $(\psi_\alpha, 0)^t$, whose free Lagrangian is given by $\frac{1}{2}\,\overline{\psi}_M(i\partial_\mu\gamma^\mu - m)\,\psi_M$. In terms of ψ_M, the relevant Yukawa coupling can be written as $-gA\psi^2 + h.c. = gA\bar{\psi}_M\frac{1+\gamma_5}{2}\psi_M + h.c.$ According to the discussion in Chapter 6, the propagator of Majorana particle ψ_M is just the same as the case of ordinary Dirac fermion, except that not only $\psi_M\overline{\psi}_M$ but also $\psi_M\psi_M^t$ type propagator exist, due to the fact $(\psi_M)^C = \psi_M$ (see (6.40) and (6.41)). Paying attention to this specific feature of the Majorana particle, we may calculate Fig. 7.2(b) according to the ordinary Feynman rule. We can directly check that there is a cancellation of the quadratic divergence between the diagrams (a) and (b). Actually, we find that for $m = 0$ the sum of these two diagrams identically vanish. We may also check that when SUSY is broken (spontaneously or explicitly) and only A gets a mass M_{SUSY}, denoting the extent of the SUSY breaking, there remains a logarithmic divergence in the sum of these diagrams, i.e. (for $m = 0$) $\propto M_{SUSY}^2 \log\Lambda$. This is why M_{SUSY} should not exceed 1 (TeV) or so, since otherwise the hierarchy problem will arise again. Let us note that another type of diagram coming from the contraction of two identical vertices, e.g. of the type of $A\bar{\psi}_M\frac{1+\gamma_5}{2}\psi_M$, does not yield the quadratic divergence, even for $m \neq 0$, and therefore has been ignored in the discussion above.

7.7 Vector superfield

Though the general superfield, discussed in section 7.4, is a complex field in general, we may get an irreducible representation V of SUSY by imposing a hermiticity condition

$$V^\dagger = V, \tag{7.67}$$

since the infinitesimal transformation of SUSY is described by an operator, $i(\theta Q + \overline{\theta Q})$, which may be regarded to be hermitian as Q and $\bar{Q}$ are differential operators. Then the component fields appearing as a coefficient of $\theta\sigma^\mu\overline{\theta}$ is a real vector field V_μ, which may be identified as a gauge field in supersymmetric gauge theories. Here we will retain in Abelian gauge theory. The generalization to non-Abelian case is rather straightforward, once we realize what are the new features of SUSY gauge theories from the simplified Abelian case, since SUSY and gauge symmetries are mutually independent symmetries. The "real" superfield may be written as follows

$$\begin{aligned} V &= C + i\theta\chi - i\overline{\theta}\overline{\chi} + \frac{i}{2}\theta\theta(M+iN) - \frac{i}{2}\overline{\theta\theta}(M-iN) \\ &- \theta\sigma^\mu\overline{\theta}V_\mu + i\theta\theta\overline{\theta}(\overline{\lambda} + \frac{i}{2}\overline{\sigma}^\mu\partial_\mu\chi) - i\overline{\theta\theta}\theta(\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}) \\ &+ \frac{1}{2}\theta\theta\overline{\theta\theta}(D - \frac{1}{2}\Box C), \end{aligned} \tag{7.68}$$

where bosonic component fields C, M, N, V_μ and D are all real. In the above equation the lower component fields $C, \chi, \bar{\chi}$ have appeared also at the positions of higher component fields with derivatives. To see the reason, let us consider a supersymmetric extension of (Abelian) gauge transformation (see (7.93))

$$V \to V' = V + i(\Lambda - \Lambda^\dagger), \tag{7.69}$$

with the transformation parameter Λ being a chiral superfield. Expand the Λ in terms of component fields as $\Lambda = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)$. Then from the "non-locality" of the coordinate y^μ,

$$\begin{aligned} i(\Lambda - \Lambda^\dagger) &= i(A - A^*) + i\sqrt{2}(\theta\psi - \overline{\theta\psi}) + i\theta\theta F - i\overline{\theta\theta}F^* \\ &- \theta\sigma^\mu\overline{\theta}\partial_\mu(A + A^*) - \frac{1}{\sqrt{2}}(\theta\theta\overline{\theta}\overline{\sigma}^\mu\partial_\mu\psi - \overline{\theta\theta}\theta\sigma^\mu\partial_\mu\overline{\psi}) \\ &- \frac{i}{4}\,\theta\theta\overline{\theta\theta}\Box(A - A^*). \end{aligned} \tag{7.70}$$

It is now easy to read off the SUSY gauge transformation of each component field

$$C \to C' = C + i(A - A^*), \tag{7.71}$$

$$\chi \to \chi' = \chi + \sqrt{2}\psi, \tag{7.72}$$

$$M + iN \to M' + iN' = M + iN + 2F, \tag{7.73}$$

$$V_\mu \to V'_\mu = V_\mu + \partial_\mu(A + A^*), \tag{7.74}$$

$$\lambda \to \lambda' = \lambda, \tag{7.75}$$

$$D \to D' = D. \tag{7.76}$$

We realize that, because of the specific way of writing in (7.68), the gauge fermion λ and the auxiliary field D are gauge invariant: (7.75) and (7.76). This property is desirable, since the gauge fermion, e.g., belongs adjoint representation, Which is gauge invariant in the Abelian gauge theory. On the other hand, by use of the transformation property (7.71) – (7.73) , one immediately know that by suitable choice of the parameters ImA, ψ, F, the lower component fields C, χ, M, N are readily gauged away. Thus, without loss of generality, we may assume the following form of the vector superfield, called Wess–Zumino gauge:

$$V = -\theta\sigma^\mu\bar{\theta}V_\mu + i(\theta\theta\overline{\theta\lambda} - \overline{\theta\theta}\theta\lambda) + \frac{1}{2}\theta\theta\overline{\theta\theta}D. \tag{7.77}$$

In some sense, the above situation is similar to what we expect in ordinary gauge theories, i.e. the longitudinal component of gauge field, which can be written as a derivative of a scalar function may be gauged away. However, we should also note that even after moving to the Wess–Zumino gauge, we still have a degree of gauge transformation due to the real part of $A(x)$,

$$V_\mu \to V'_\mu = V_\mu + 2\partial_\mu(\mathrm{Re}A), \tag{7.78}$$

which corresponds to the ordinary gauge transformation with transformation parameter ReA.

In this way, we have identified a SUSY multiplet containing vector field V_μ, called vector multiplet,

$$(V_\mu(x),\ \lambda(x),\ D(x)). \tag{7.79}$$

The mass dimensions of these component fields are readily known to be (1, 3/2, 2). Thus λ should be a physical field called gauge fermion or gaugino, the SUSY partner of gauge boson V_μ, while D is an auxiliary field, which should be written in terms of physical scalar fields once Euler–Lagrange equation for D field is used.

- **Field strength superfield**

In order to get the kinetic term for the vector multiplet, we need to construct some gauge invariant (covariant in non-Abelian case) superfield, whose product provides the desirable kinetic term, out of gauge variant superfield V.

We have learned from the aforementioned gauge transformation property that the gauge fermion λ is gauge invariant, while V_μ transforms inhomogeneously. Thus a reasonable way to get the gauge invariant superfield is to differentiate V, so that it can extract the λ_α field as the lowest component:

$$\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\bar{\theta}}\frac{\partial}{\partial\theta^\alpha}\ V. \tag{7.80}$$

This is not a correct way of doing, since the differential operators $\frac{\partial}{\partial\bar{\theta}}$ etc. do not commute with SUSY algebra, and some modification is needed:

$$W_\alpha = -\frac{1}{4}\ \overline{D}\overline{D}D_\alpha\ V. \tag{7.81}$$

Such defined W_α and $\overline{W}_{\dot{\alpha}}$ are called field strength superfield.

The field strength superfields turn out to be chiral superfields. In fact, we can easily show, for instance,

$$\bar{D}_{\dot{\beta}}\ W_\alpha = -\frac{1}{4}\bar{D}_{\dot{\beta}}\overline{D}\overline{D}D_\alpha V = 0, \tag{7.82}$$

where we have used the property $\bar{D}_{\dot{\beta}}\overline{D}\overline{D} = 0$ as the result of $\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0$. Another important feature of W_α is that it is gauge invariant:

$$\begin{aligned} W_\alpha \to W'_\alpha &= W_\alpha - \frac{i}{4}\ \overline{D}\overline{D}D_\alpha\ (\Lambda - \Lambda^\dagger) = W_\alpha - \frac{i}{4}\ \overline{D}\{\overline{D}, D_\alpha\}\ \Lambda \\ &= W_\alpha - \frac{i}{4}\ \{\overline{D}, D_\alpha\}\overline{D}\Lambda = W_\alpha, \end{aligned} \tag{7.83}$$

where we have invoked to the chiral nature of Λ and $\bar{\Lambda}$, $\bar{D}_{\dot{\alpha}}\Lambda = D_\alpha\Lambda^\dagger = 0$.

Let us power expand W_α over θ, in such a way where component fields are functions of y^μ. For such purpose we first rewrite V in terms of $y^\mu = x^\mu + i(\theta\sigma^\mu\bar{\theta})$:

$$\begin{aligned} V &= -\theta\sigma^\mu\overline{\theta}V_\mu(y) + i(\theta\theta\overline{\theta}\overline{\lambda}(y) - \overline{\theta}\overline{\theta}\theta\lambda(y)) \\ &+ \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}(D(y) + i\ \partial_\mu V^\mu(y)). \end{aligned} \tag{7.84}$$

We also rewrite "SUSY-covariant" derivatives (7.44), (7.45) in terms of y^μ and $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \tag{7.85}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}. \tag{7.86}$$

Then, according to the definition (7.81), W_α is obtained after some calculus to be

$$\begin{aligned} W_\alpha &= -i\lambda_\alpha(y) + [\delta_\alpha{}^\beta D(y) - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta(\partial_\mu V_\nu(y) - \partial_\nu V_\mu(y))]\theta_\beta \\ &+ \theta\theta(\sigma^\mu)_{\alpha\dot{\alpha}}\ \partial_\mu\bar{\lambda}^{\dot{\alpha}}(y). \end{aligned} \tag{7.87}$$

As we expected the lowest component is the gauge fermion λ and all of λ, D and the field strength $F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu$ are gauge invariant.

- **Kinetic term for the gauge field**

¿From the above expression for W_α, it is clear that $W^\alpha W_\alpha|_F$ provides the kinetic term $-i\lambda\sigma^\mu\partial_\mu\bar{\lambda}$, etc. for the members of gauge multiplet:

$$\begin{aligned}\mathcal{L}_{gauge-kin} &= \frac{1}{4}\left(W^\alpha W_\alpha|_F + h.c.\right) \\ &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}} + \frac{1}{2}D^2. \end{aligned} \tag{7.88}$$

The kinetic term for gauge boson V_μ is of ordinary type. The kinetic term for the gauge fermion λ may be understood as that for a Weyl fermion. Since the gauge fields should be "real", it may be natural to expect that gauge fermion is described by a Majorana fermion $\lambda_M = (\lambda_\alpha, \bar{\lambda}^{\dot{\alpha}})^t$. In fact, the kinetic term can be re-written as $\frac{1}{2}\bar{\lambda}_M i\partial\!\!\!/\lambda_M$, up to a total derivative. (A Weyl and a Majorana fermions are identical, as far as the fermions are massless.) As we expected the "kinetic term" for the D field does not have any derivative and D should be understood as an auxiliary field.

- **Gauge invariant Lagrangian**

Let us consider a SUSY $U(1)$ gauge theory, whose typical example is SUSY QED. Generalization to the non-Abelian case is rather straightforward.

Suppose we have chiral superfields ϕ_i, which have electric charges Q_i. Under a global $U(1)$ gauge transformation the chiral superfield transform as

$$\phi_i \to \phi_i' = e^{-i2eQ_i\Lambda}\ \phi_i, \tag{7.89}$$

where gauge transformation parameter Λ should be regarded as a chiral superfield, not to spoil the chirality of ϕ by the transformation. Just as in ordinary field theory, the self interaction of chiral superfields, i.e. superpotential should be invariant under the transformation (7.89):

$$W(\phi_i) \ \to \ W(\phi_i') = W(\phi_i). \tag{7.90}$$

When the $U(1)$ transformation is made local, allowing Λ to depend on y^μ, the superpotential clearly remains invariant. The kinetic term for ϕ_i, however, is no longer invariant,

$$\phi_i^\dagger\phi_i \ \to \ e^{-2ieQ_i(\Lambda-\Lambda^\dagger)}\ \phi_i^\dagger\phi_i. \tag{7.91}$$

In order to make it gauge invariant we now introduce the vector superfield V to compensate the factor $e^{-2ieQ_i(\Lambda-\Lambda^\dagger)}$, i.e. the kinetic term is now replaced by "covariant derivative" term,

$$\phi_i^\dagger e^{2eQ_iV}\phi_i, \tag{7.92}$$

where V should transform under the local gauge transformation inhomogeneously,

$$V \ \to \ V' = V + i(\Lambda - \Lambda^\dagger). \tag{7.93}$$

Actually the Taylor expansion of e^{2eQ_iV} ends at V^2: $V^2 = \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}V_\mu V^\mu$, $V^n = 0$ for $n \geq 3$. By use of this fact we can check that the SUSY invariant (up to a total derivative) D-term of (7.92) really contains the ordinary covariant derivatives for scalars and fermions, though it also contain interaction terms of $\lambda, \bar{\lambda}, D$, which are characteristic to the SUSY gauge theory:

$$\begin{aligned}\phi_i^\dagger e^{2eQ_iV}\phi_i|_D &= (D_\mu A_i)^*(D^\mu A_i) - i\bar{\psi}_i\bar{\sigma}^\mu D_\mu\psi_i + |F_i|^2 \\ &- \sqrt{2}ieQ_i(A_i\bar{\lambda}\bar{\psi}_i - A_i^*\lambda\psi_i) + eQ_iD|A_i|^2, \qquad (7.94)\end{aligned}$$

where D_μ denotes the ordinary gauge-covariant derivatives

$$D_\mu A_i = (\partial_\mu + ieQ_iV_\mu)A_i, \quad D_\mu\psi_i = (\partial_\mu + ieQ_iV_\mu)\psi_i. \qquad (7.95)$$

The terms in the second line of (7.94) are the new types of gauge interactions due to the presence of gaugino $\lambda, \bar{\lambda}$ and auxiliary field D, whose coupling constants are identical, i.e. e, because of the SUSY.

7.8 SUSY QED

As the immediate application of the SUSY $U(1)$ gauge theory discussed above, let us consider the supersymmetric extention of QED, namely SUSY QED.

The matter field, electron, should be identified with the fermionic components of two (right-handed) chiral superfields,

$$\phi_- = (\tilde{e}_R, e_R^-, F_R), \quad \phi_+ = (\tilde{e}_L^*, e_R^+, F_L^*), \qquad (7.96)$$

where e_R^- and $e_R^+ = (e_L^-)^C$ are the right-handed components of electron and positron in 2-component notation, and by $\tilde{e}_{R,L}$ we denote the SUSY partners of $e_{R,L}^-$, "selectron". The vector superfield $V = (V_\mu, \lambda, D)$ contains the photon field V_μ and its superpartner, "photino", λ.

Since the electron has a bare mass term $-m\bar{e}e$ in ordinary QED, we employ the following superpotential

$$W = m\phi_-\phi_+. \qquad (7.97)$$

Combining with the SUSY invariant and gauge invariant kinetic terms for vector and chiral superfields, we get the full Lagrangian for the SUSY QED

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}(W^\alpha W_\alpha|_F + h.c.) + \phi_-^\dagger e^{-2eV}\phi_-|_D + \phi_+^\dagger e^{2eV}\phi_+|_D \\ &- (m\phi_-\phi_+|_F + h.c.). \qquad (7.98)\end{aligned}$$

Writing explicitly in terms of the component fields, the Lagrangian reads as

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2}D^2$$

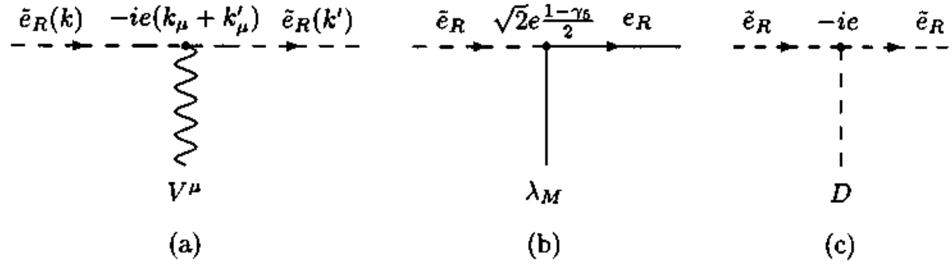


Fig. 7.3

$$
\begin{aligned}
&+ \quad (D_\mu \tilde{e}_R)^*(D^\mu \tilde{e}_R) + (D_\mu \tilde{e}_L^*)^*(D^\mu \tilde{e}_L^*) \\
&- \quad i\overline{e_R^-}\bar{\sigma}^\mu D_\mu e_R^- - i\overline{e_R^+}\bar{\sigma}^\mu D_\mu e_R^+ + |F_R|^2 + |F_L|^2 \\
&+ \quad \sqrt{2}ie(\tilde{e}_R\bar{\lambda}\overline{e_R^-} - h.c.) - \sqrt{2}ie(\tilde{e}_L^*\bar{\lambda}\overline{e_R^+} - h.c.) - eD(|\tilde{e}_R|^2 - |\tilde{e}_L|^2) \\
&+ \quad m(e_R^- e_R^+ + h.c.) - m(\tilde{e}_R F_L^* + \tilde{e}_L^* F_R + h.c.). \qquad (7.99)
\end{aligned}
$$

So far spinors for electron and positron have been all written down by use of the 2-component notation. When we calculate Feynman diagrams, however, it will be more convenient to utilize the familiar 4-component notation, so that we can use the ordinary form for the spinor propagators. Thus we define 4-component spinors, Dirac spinor e denoting the electron, and a Majorana spinor λ_M for the photino:

$$
e = \begin{pmatrix} e_{R\alpha}^- \\ e_L^{-\dot{\alpha}} \end{pmatrix}, \qquad (7.100)
$$

$$
\lambda_M = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix}. \qquad (7.101)
$$

In terms of these 4-component spinors, the Lagrangian is given as

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}\overline{\lambda_M}\partial\!\!\!/\lambda_M + \frac{1}{2}D^2 \\
&+ \quad |D_\mu \tilde{e}_R|^2 + |D_\mu \tilde{e}_L|^2 + i\bar{e}D\!\!\!\!/\,e + |F_R|^2 + |F_L|^2 \\
&+ \quad \sqrt{2}ie\{(\tilde{e}_R^*\overline{\lambda_M}\frac{1+\gamma_5}{2}e + \tilde{e}_L^*\overline{\lambda_M}\frac{1-\gamma_5}{2}e) - h.c.\} - eD(|\tilde{e}_R|^2 - |\tilde{e}_L|^2) \\
&- \quad m\bar{e}e - m(\tilde{e}_R^* F_L + \tilde{e}_L^* F_R + h.c.). \qquad (7.102)
\end{aligned}
$$

As we clearly see in the Feynman rule shown in Fig. 7.3 the coupling constants of the photon, photino and the auxiliary fields are the same, i.e. e.

After eliminating the auxiliary fields $F_{R,L}$ and D, the scalar potential gets contributions from both of superpotential and gauge interaction. To see this explicitly, let us write down the terms including the auxiliary fields in the Lagrangian

$$
|F_R|^2 + |F_L|^2 - m(\tilde{e}_R F_L^* + \tilde{e}_L^* F_R + h.c.) + \frac{1}{2}D^2 - eD(|\tilde{e}_R|^2 - |\tilde{e}_L|^2). \qquad (7.103)
$$

By use of equations of motion for the auxiliary fields

$$\begin{aligned} F_R &= m\tilde{e}_L, \quad F_L = m\tilde{e}_R, \\ D &= e(|\tilde{e}_R|^2 - |\tilde{e}_L|^2), \end{aligned} \tag{7.104}$$

we get the scalar potential

$$\begin{aligned} V &= |F_R|^2 + |F_L|^2 + \frac{1}{2}D^2 \\ &= m^2(|\tilde{e}_R|^2 + |\tilde{e}_L|^2) + \frac{1}{2}e^2(|\tilde{e}_R|^2 - |\tilde{e}_L|^2)^2. \end{aligned} \tag{7.105}$$

We easily find that the vacuum state of this theory is achieved by vacuum expectation values $\langle\tilde{e}_R\rangle = \langle\tilde{e}_L\rangle = 0$. The corresponding vacuum energy $E_v = \langle 0|H|0\rangle$ also vanishes, and it indicates that SUSY is not spontaneously broken, as we will see in the following section. The term $m^2(|\tilde{e}_R|^2 + |\tilde{e}_L|^2)$ in the potential V thus provides scalar mass-squared m^2. This means the masses of electron and selectron are the same as the inevitable consequence of SUSY.

7.9 SUSY Yang-Mills theories

The argument for SUSY $U(1)$ gauge theory is easily generalized to non-Abelian gauge theory including the SUSY extension of the standard model, such as MSSM.

The transformation of a set of chiral superfields, denoted by a column vector ϕ, belonging to a representation of non-Abelian Yang-Mills gauge group, takes a form

$$\phi \to \phi' = e^{-i2g\Lambda}\ \phi, \tag{7.106}$$

where Λ is a matrix belonging to the Lie algebra

$$\Lambda = T^a\Lambda^a, \tag{7.107}$$

where Λ^a are transformation parameters and the generators T^a satisfy (in a suitable normalization)

$$[T^a, T^b] = if^{abc}T^c, \quad Tr\ (T^aT^b) = \frac{1}{2}\,\delta^{ab}. \tag{7.108}$$

Correspondingly the vector superfields V^a belonging to the adjoint representation of the group behave as

$$e^{2gV} \to e^{2gV'} = e^{-i2g\Lambda^\dagger}e^{2gV}e^{i2g\Lambda} \tag{7.109}$$

where

$$V = T^aV^a, \ V' = T^aV'^a, \tag{7.110}$$

so that the term $\phi^\dagger e^{2gV}\phi$ becomes gauge invariant.

A gauge covariant field strength superfield is obtained by a straightforward extension of the Abelian case (7.81),

$$W_\alpha = -\frac{1}{4}\,\overline{DD}e^{-2gV}\,D_\alpha\,e^{2gV}. \tag{7.111}$$

It is easy to show that such defined field strength has a desirable transformation rule,

$$W_\alpha \to W'_\alpha = e^{-i2g\Lambda}W_\alpha e^{i2g\Lambda}. \tag{7.112}$$

Thus gauge invariant and SUSY invariant Yang-Mills theory with a gauge invariant superpotential $W(\phi)$ is described by

$$\begin{aligned}\mathcal{L} &= \frac{1}{8g^2}\mathrm{Tr}(W^\alpha W_\alpha|_F + h.c.) + \phi^\dagger e^{2gV}\phi|_D \\ &- (W(\phi)|_F + h.c.).\end{aligned} \tag{7.113}$$

The generalization to the case with several chiral superfields is quite straightforward. Let us note that we still can impose a condition of Wess–Zumino gauge for each of V^a. The Feynman rule shown in Fig. 7.3 for SUSY QCD is now modified in a trivial manner by a prescription,

$$-ie \;\to\; igT^a, \tag{7.114}$$

for the gauge interaction vertices of $(V^a_\mu, \lambda^a_M, D^a)$, with $(\tilde{e},\ e)$ being replaced by the scalar and spinor components of ϕ.

7.10 Minimal Supersymmetric Standard Model (MSSM)

The SUSY Yang-Mills theory of the most physical interest is the SUSY extension of the standard model. Its minimal version, with minimal number of Higgs chiral multiplets, is called Minimal Supersymmetric Standard Model (MSSM). In this section, we discuss MSSM very briefly, focusing on the basic structure of the model. For more detailed discussion on the MSSM and its various phenomenological implications, we ask readers to refer to the nice reviews in the literature (Haber and Kane, 1985; Weinberg, 2000).

Basically the construction of the MSSM is achieved just following the argument of the previous section. In particular the Lagrangian is given by (7.113), though the second term, the kinetic term (with covariant derivative) of chiral field ϕ, should be replaced by the sum of the kinetic term of each existing chiral field with distinct gauge generator $V = T^aV^a$ corresponding to its reperesentation of the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. The superpotential W of the chiral superfields may take, in principle, the most general form compatible with gauge symmetry, as in the standard model. We will argue, however, that some restriction is needed, since otherwise some serious problems, such as too rapid proton decay is induced. The restriction is due to a global symmetry called R-symmetry, as we will see below.

The gauge group of MSSM $SU(3)_c \times SU(2)_L \times U(1)_Y$ is just the same As that of the standard model, and the gauge bosons and gauge fermions form vector mmmultiplets, whose interactions are uniquely determined by gayge principle, as is seen in (7.113). The matter fields, on the other hand, are of our choice. In the stabdard model, the quarks and leptons to start with are Weyl fermions. Thus it is natural to assign quarks and leptons as the members of chiral multiples. The remaining matter fields, Higgs, being complex fields, should belong to a chiral multiplet.

Actually we realize that to construct a realistic theory, one $SU(2)_L$ doublet of Higgs chiral superfields is not enough, and we should add another doublet. The reason is two-fold.

(1) In the (non-SUSY) standard model the Yukawa couplings or masses of down-type and up-type quarks are provided by ϕ and $\tilde{\phi} \equiv i\sigma_2\phi^*$, respectively. One may naively expect that the Yukawa couplings are just replaced by the corresponding superpotential of the chiral superfields of quarks and the Higgs doublet. Unfortunately, its does not work. This is because, $\tilde{\phi}$ is obtained by the operation of complex conjugation (charge conjugation), and therefore the chiralities of ϕ and $\tilde{\phi}$ are opposite, while superpotential should be the polynomial of superfields with the same chirality. We thus need to introduce two independent Higgs doublets in order to provide masses to down-type and up-type quarks, say H_D and H_U, with the same representations of the gauge group as those of ϕ and $\tilde{\phi}$, both being left-handed chiral superfields.

(2) Suppose we introduce chiral superfields whose scalar components are the members of one Higgs doublet. As the new feature of the SUSY theory, chiral superfields also contain the fermionic superpartners of the Higgs scalars, i.e. Higgs fermions. As the Higgs fermions are Weyl fermions, they yield triangle gauge anomalies. For example "$U(1)^3_Y$" anomaly, coming from the triangle diagram where all three vertices are made of the currents coupled with the $U(1)_Y$ gauge boson, arises by the presence of the Higgs fermion. Ther also appear the $(SU(2)_L)^2U(1)_Y$ type anomaly as well. Such anomalies will be vancelled by the introduction of another Higgs doublet, just because they have just opposite quantum numbers, with weak-hypercharges $Y = \pm 1$. For instance $U(1)^3_Y$ anomaly will vanish as $(+1)^3 + (-1)^3 = 0$.

Thus the matter fields of MSSM with 3 generations of quarks and leptons are all given as left-handed chiral superfields

$$\begin{aligned} Q_i &= \begin{pmatrix} u_i \\ d_i \end{pmatrix}, \quad \bar{u}_i, \quad \bar{d}_i; \quad L_i = \begin{pmatrix} \nu_{li} \\ l_i \end{pmatrix}, \quad \bar{l}_i \quad (i = 1, 2, 3) \\ H_D &= \begin{pmatrix} -\varphi_1^0 \\ \varphi_1^- \end{pmatrix}, \qquad H_U = \begin{pmatrix} \varphi_2^+ \\ \varphi_2^0 \end{pmatrix}, \end{aligned} \tag{7.115}$$

where we have used the same letters to denote the chiral superfields as those of the ordinary particles in the standard model. For instance for generic quark q and Higgs

H chiral superfields, their spin $s = 0, 1/2$ components together with the auxiliary fields F_q, F_H are written as

$$q = (\tilde{q}, q, F_q), \qquad H = (H, \tilde{H}, F_H), \tag{7.116}$$

where $\tilde{q}$ and $\tilde{H}$ are the superpartners of ordinary particles, i.e. a squark and a Higgs fermion.

Next thing to do for the construction of the MSSM is to choose the form of the superpotential W. To provide the Yukawa couplings necessary to give quark and lepton masses and the self-interaction of Higgs doublets, we employ the following superpotential

$$\begin{aligned} W &= f^u_{ij}\bar{u}_i Q_j H_U + f^d_{ij}\bar{d}_i Q_j H_D + f^l_{ij}\bar{l}_i L_j H_D \\ &+ \mu H_U H_D, \end{aligned} \tag{7.117}$$

where $i = 1, 2, 3$ is the generation index, and the abbreviation $Q_j H_U$ etc. is used for $Q^t_j(i\sigma_2)H_U$ etc., to denote the $SU(2)$ invariant products of two doublets.

Actually this superpotential is not the most general one compatible with gauge symmetry. Let us note that in the standard model, the Higgs potential is the most general one, so that it can prepare every counterterms for possible UV-divergences at the quantum level. It is important to note that in the supersymmetric standard model, the lepton doublet L and the Higgs doublet H_D have exactly the same quantum number and a priori there is no way to distinguish these two, though in the non-SUSY standard model, they were clearly discriminated by the difference of their spins. In fact, the above superpotential has not included, for instance, the following type of possible gauge invariant operators

$$\bar{u}\bar{d}\bar{d}, \;\; \bar{d}QL, \;\; \bar{l}L^2, \tag{7.118}$$

These terms clearly break either of baryon or lepton number, B or L, and will easily lead to too rapid B or L violating processes such as proton decay, or too frequent FCNC processes such as $\mu \to e\gamma$, as the coefficients of these operators are generally independent of the Yukawa couplings given in (7.117) and can be new sources of flavor mixings. Thus it is desirable to devise a mechanism to guarantee the absence of these undesirable operators.

Generally, to ignore terms, or operators, which is compatible with gauge symmetry is not allowed for the theory to be renormalizable, since such terms are gauge invariant and they are generally induced at the quantum level with UV-divergence and the counterterms are needed anyway. Thus the elimination of the terms listed up in (7.118) does not seem to be justified. (One may expect that the "non-renormalization theorem", characteristic to the SUSY theories, may work as the justification. The SUSY, however, should be broken eventually for the theory to be realistic.) The elimination is, at least, technically allowed if the elimination is able to be regarded as the consequence of some global symmetry, independent of the local gauge symmetry. It is not hard to imagine that the terms incompatible with the global symmetry, imposed on the whole Lagrangian, will never appear at any orders of quantum corrections. Let us note that while the tri-linear term in the superpotential $\bar{d}QH_D$ yields ordinary Yukawa coupling of ordinary particles $\bar{d}QH_D$, undesirable term $\bar{d}QL$ will result in a tri-linear coupling with one superpartner and two ordinary particles, $\bar{d}Q\tilde{L}$. Thus in order to distinguish these two types of terms, it seems to be natural to devise a global symmetry, which assign different transformation properties for ordinary particles and their superpartners. We thus consider R-symmetry, which is a characteristic symmetry in SUSY theories.

What we consider is some global $U(1)$ symmetry. It is trivial that to distinguish ordinary particles and their superpartners, the global symmetry should assign different $U(1)$ charges to the different members of the same chiral multiplet. Hence a global symmetry which assign an overall phase for the all members of a chiral multiplet is not enough for our purpose. R-symmetry is the symmetry under the chiral transformation of the Grassmannian coordinates of superspace,

$$\theta \to \theta' = e^{i\varphi}\theta, \quad \bar{\theta} \to \bar{\theta}' = e^{-i\varphi}\bar{\theta}, \tag{7.119}$$

or

$$\begin{pmatrix} \theta \\ \bar{\theta} \end{pmatrix} \;\to\; e^{i\varphi\gamma_5}\begin{pmatrix} \theta \\ \bar{\theta} \end{pmatrix}. \tag{7.120}$$

Accordingly a generic chiral superfield ϕ transforms as

$$\phi = A + \sqrt{2}\theta\psi + \theta\theta F \to \phi' = e^{ic\varphi}\phi = A' + \sqrt{2}\theta'\psi' + \theta'\theta' F', \tag{7.121}$$

where c denotes the overall $U(1)$ charge of the chiral multiplet. The each component field therfore transforms as

$$A' = e^{ic\varphi}, \; \psi' = e^{i(c-1)\varphi}\psi, \; F' = e^{i(c-2)\varphi}F. \tag{7.122}$$

The F-term of the superpotential $W|_F \sim \frac{\partial^2}{\partial\theta^2}W$ should be invariant under the transformation of R-symmetry. Thus the transformation property of the superpotential should be

$$W \;\to W' = e^{2i\varphi}W. \tag{7.123}$$

For instance, for the superpotential $W = \phi_1\phi_2\phi_3$ of some three chiral superfields to be R-symmetric, their $U(1)$ charges should satisfy $c_1 + c_2 + c_3 = 2$. Let us note that, since the R-symmetry is a chiral symmetry, other terms in the Lagrangian, i.e. kinetic terms with covariant derivatives are automatically R-symmetric, though the gauge fermions transform non-trivially.

We will assign the $U(1)$ charge $c = 1$ for all of quark and lepton chiral superfields, $Q, \bar{u}, \bar{d}, L, \bar{l}$, and $c = 0$ for Higgs chiral superfields H_D and H_U. By taking a specific value $\varphi = \pi$, we can define a discrete symmetry, "R-parity". The (overall) parity of quark and lepton superfields is -1, while that of Higgs superfields is 1. Now the condition reduce, with $\varphi = \pi$, to

$$W' = W, \quad \sum_i c_i = 0 \ (\mathrm{mod}\ 2). \tag{7.124}$$

It is now easy to see that the all terms in (7.118) do not satisfy the above condition, while the terms in (7.117) are all consitent with the condition.

For each component field of the chiral multiplets the assigned $U(1)$ phases for $\varphi = \pi$, i.e. R-parity denoted by R, is given by a compact form,

$$R = (-1)^{2J}(-1)^{c} = (-1)^{2J}(-1)^{3B+L}, \tag{7.125}$$

where J is the spin of the field and B, L are baryon and lepton numbers, and $3B + L = 1$ for quarks and leptons and 0 for Higgs doublets. We thus easily know that all of the quarks, leptons and Higgs have $R = 1$ while their superpartnes all have $R = -1$. In this way we have suceeded to distinguish ordinary particles and their superpartners. In addition, the absence of the undesirable terms in (7.118) is justified as the consequence of the R-parity symmetry. We further note that, in general, hevier superpartners having odd R-parity should decay into lighter superpartners and other ordinary particles, but not into a final state composed only of ordinary particles, which have even R-parity. Namely the R-parity invariance enforces the lightest supersymmetric particle (LSP) to be absolutely stable, thus being an interesting candidate of (cold) dark matter.

Though we have implicitly assumed that the SUSY is an exact symmetry, actually the SUSY should be broken spontaneously (see section 7.12) or explicitly (with "soft" terms, whose coefficients have positive mass dimensions, in order not to spoil the nice property of the quadratic divergence cancellation), since the superpartners of ordinary quarks and leptons having the same masses as those of the ordinary particles have not been discovered. The theory should also possess the spontaneous gauge symmetry breaking. These two kinds of symmetry breaking cause the mixings among various particles via their mass(-squared) terms, in various sectors, which makes the complete analysis of MSSM (Haber and Kane, 1985; Weinberg, 2000) rather cumbersome.

We finally briefly comment on the specific feature of MSSM in its Higgs sector. After the elimination of the auxiliary fields $F_{H_{U,D}}$, the quadratic terms of Higgs bosons in the obtained scalar potential are $|\mu|^2(H_D^\dagger H_D + H_U^\dagger H_U)$. This in turn means that the spontaneous gauge symmetry breaking does not happen, as long as SUSY is preserved. An intriguing mechanism to achieve the negative mass-squared of Higgs boson, necessary for the symmetry breakdown, has been proposed (Inoue, et al., 1982), where the negative mass-squared is realized by the SUSY breaking and the quantum correction due to the loop of a heavy fermion with large Yukawa coupling, i.e. t quark.

We also note that the (Higgs scalar)^4 term of the scalar potential is uniquely provided by the D-term contribution, e.g. $\frac{g'^2}{8}(H_D^\dagger H_D - H_U^\dagger H_U)^2$ which is obtained after the elimination of D-auxiliary field of $U(1)$ vector multiplet. Thus the coefficient of the quartic term is completely determined by the gauge coupling constants and is of the order of g^2 or g'^2. This means that the Higgs masses are comparable to M_Z or M_W, provided SUSY is an exact symmetry. Though actually the SUSY breaking mass-squared terms for the Higgs bosons considerably modify the range of the Higgs masses, there should remain at least one light physical Higgs boson, whose mass $\leq M_Z$. Intuitively, this is because at the limit of $M_{SUSY} \to \infty$, the theory should reduce to the non-SUSY standard model, where one physical Higgs, whose mass is independent of M_{SUSY}, should exist.

7.11 Some Phenomenological Predictions of SUSY Gauge Theories

In this section we will discuss very briefly how we can search for the trace of SUY gauge theories for elementary particles and/or how we can test the characteristic phenomenological predictions of SUSY gauge theories.

Very successful prediction of the SUSY version of the standard model, Minimal Supersymmetric Standard Model (MSSM), is the gauge coupling unification. In Grand Unified Theories (GUT), which unify all interactions (except gravity) of elementary particles, three gauge couplings of $SU(3)_c$, $SU(2)_L$, $U(1)_Y$, denoted by g_3, g_2, g_1 for suitably normalized gauge generators, should be unified at higher energies. Since these couplings are those for "strong", "weak" interactions etc., such couplings do not seem to be unified at least at low energy processes. Actually the "asymptotic freedom" of non-Abelian gauge theories, and the "asymptotic non-free" nature of Abelian gauge theory, briefly discussed in chapter 5, implied by the renormalization group equations to discribe the energy evolution of gauge couplings, make the grand unification possible. Since the dependence of the gauge couplings on the energy is very mild, depending only logarithmically, the mass (energy) scale of the grand unification M_{GUT} is quite high. In the prototype (non-SUSY) $SU(5)$ GUT (Georgi and Glashow, 1974) $M_{GUT} = \mathcal{O}(10^{15})(GeV)$. Such tremendously high mass scale makes the life time of proton decay, typically predicted by GUT, quite long, assuring the (approximate) stability of atoms.

The simplest $SU(5)$ GUT model, however, seems to be facing the following difficulties. (a) First, it predicts the proton life time $\tau_P \simeq 3 \times 10^{31}$(years), much longer the age of our universe. Nevertheless, the prediction has been ruled out by the recent (Super-)Kamiokande result on the decay mode $p \to \pi^0 + e^+$. (b) Second, the precision measurement of the gauge couplings g_3, g_2, g_1 at $E \sim 10^2(GeV)$ at LEP experiments (CERN) has revealed the fact that these couplings do not meet with each other at an unique value of higher energy. It is quite impressive that such difficulties can be evaded (though there still may remain some problem) in SUSY SU(5) GUT (Sakai, 1981; Dimopoulos and Georgi, 1981), whose low-energy effective theory is MSSM. Namely in the SUSY GUT, the evolution of gauge couplings, or the β-functions in the renormalization group equations are modified by the presence of the newly introduced superpartners of ordinary particles (mainly by the gauge fermions). If the SUSY breaking mass scale M_{SUSY} is in a reasonable range, $M_{SUSY} \sim 1(TeV)$, all of such modified three "running" couplings meet at a unique value of the energy ! Furthermore, the modification of the evolution raises the unification scale a little bit, $M_{GUT} \simeq 10^{16}(GeV)$. This little change, however, considerably raises the proton life time τ_P (since $\tau_P \propto M_{GUT}^4$), thus evading the lower bound on τ_P imposed by (Super-)Kamiokande experiment. It may be worth noting that the most recent result on the lower bound of τ_P from Super-Kamiokande experiment is quite close to the prediction of the SUSY GUT, even though they still seem to be mutually compatible.

Probably the most direct confirmation of SUSY will be the discovery of the characteristic new heavy particles, i.e. the superpartners, in accelerator experiments, and extensive efforts have been made in such direction. Such superpartners should affect the low-energy physics as well, and their contributions may be (indirectly) tested by use of various low energy observables, whose probabilities have been measured precisely. In particular the SUSY breaking masses of the order M_{SUSY} of these superpartners can be arbitrary a priori, as long as they are gauge invariant. Therefore they may potentially spoil the successful predictions of the (non-SUSY) standard model, unless there are some guiding principle to control them.

For illustrative purpose, here we will briefly discuss the contributions of superpartners of quarks and leptons (squarks and sleptons) to the observables, which are rendered to be small by global symmetries, i.e. (1) $\Delta\rho$ and (2) Flavor Changing Neutral Current (FCNC). These are quantities discussed in detail in chapters 8 and 9, respectively, and are handled by global symmetries, i.e. "custodial" and flavor (or horizontal) symmetries. We will see in chapters 8 and 9, that because of these global symmetries, these observables exactly vanish at the tree level. Though the observables are induced at the quantum (loop) level, they are automatically finite and are suppressed. Thus the smallness of these quantities are guaranteed without any tuning of parameters of the theory, the property we call "natural". If SUSY is exact, i.e. if there is no SUSY breaking, the situation essentially does not change, and natural suppression of these observables are still operative. Once SUSY is broken, which is necessary anyway for the models to be realistic, the situation remarkably change, generally speaking. In particular, as we will argue below, FCNC may get into trouble, since the SUSY breaking mass terms, being flavor or generation dependent, break the flavor symmetry. Thus unless there is some guiding principle to control the mass terms, the natural suppression of FCNC is spoilt, and it may cause a serious phenomenological problem of the model.

(1) $\Delta\rho$

The ρ-parameter is defined in (4.105) and (8.5) as the ratio of charged and neutral weak gauge boson mass-squared. $\rho = 1$ at the tree or classical level because of the $SU(2)_V$ (or $SU(2)_L$) global "custodial" symmetry which remain in the Higgs sector even after the spontaneous symmetry breakdown (see 8.4). However, the symmetry is broken in the entire Lagrangian by e.g. the mass splitting between the members of $SU(2)_L$ doublet, such as $(t, b)^t$, the parameter deviates from 1, i.e. $\Delta\rho \equiv \rho - 1 \neq 0$. The contribution of the heavy top quark to $\Delta\rho$ turns out to behave as $\frac{m_t^2}{M_W^2}$. Then, one may naively expect that the superpartners, which are supposed to be heavy, $\sim 1(TeV)$ or so, may strongly affect $\Delta\rho$ and SUSY gauge theories may contradict with the experimental upper bound on $\Delta\rho$. Fortunately the SUSY breaking mass-squared given for the doublet of squarks, for instance $(\tilde{u}, \tilde{d})^t$, does not break the global symmetry. Namely the $SU(2)_L$ gauge invariance makes the SUSY breaking mass-squared term for the squarks $M_{SUSY}^2(|\tilde{u}|^2 + |\tilde{d}|^2)$ invariant under the transformation of the custodial $SU(2)$ symmetry. Thus we expect the contribution of the squark doublet is under control. In fact we will see in (8.18) that the contribution is strongly suppressed by the factor $(m_u^2 - m_d^2)^2/(M_W^2 M_{SUSY}^2)$ (the mixing between the superpartners of left-handed and right-handed quarks has been ignored for brevity). This kind of suppression is generally stated as the "decoupling" phenomena of heavy particles, which we will systematically analyze in section 8.2. Thus the newly added contributions, characteristic to SUSY gauge models, are not troublesome concerning $\Delta\rho$, as long as M_{SUSY} is not too small.

(2) FCNC

We will discuss in section 9.2 that to guarantee the natural suppression of FCNC processes, whose rates are experimentally stringently bound and these processes are often called rare processes, fermions of the same electric charge and chirality should belong to the same representation of the gauge group (Glashow-Weinberg's condition). Since the SUSY generators commute with those of gauge symmetry, these two symmetries are mutually independent. Thus the superpartners should belong to the same representations as those of ordinary quarks and leptons, and the Glashow-Weinberg's condition is still satisfied in SUSY extension of the gauge theories satisfying the condition. In fact, each of neutral currents coupled to Z weak gauge boson in quark and squark sectors, for instance, is flavor-diagonal and does not have FCNC. There appear, however, new type of neutral current in SUSY gauge theories, namely the (fermionic) current coupled with gauge fermion, which is bi-linear form of a fermion and its superpartner, of the type shown in Fig. 7.3(b). As we will see below if SUSY breaking mass-squared for superpartners are flavor-dependent, there arises a mismatching between the mass(-squared) matrices of fermions and superpartners, and when we move to their mass-eigenstates the neutral current coupled to the gauge fermion may have FCNC. This happens already at the SUSY QED discussed in 7.8 once the model discussed there is generalized so that it includes (three) generations of leptons. Thus leptonic FCNC process, for instance $\mu \to e\gamma$, becomes possible via the exchange of the gauge fermion, photino λ_M, as shown in Fig. 7.4.

We are now ready to confirm the above statement by a little explicit calculation

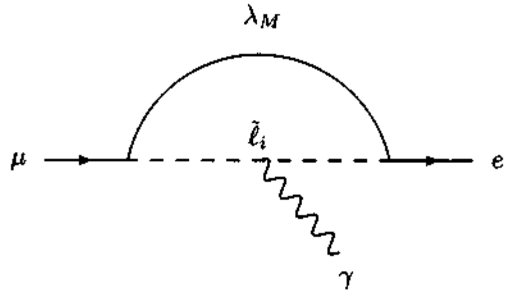


Fig. 7.4

in the SUSY QED with three generations of leptons. The chiral multiples in (7.96) is generalized to include three generations, $\alpha = 1,2,3$, as

$$\phi_{-\alpha} = (\tilde{l}_{\alpha R}, l^-_{\alpha R}, F_{\alpha R}), \quad \phi_{+\alpha} = (\tilde{l}^*_{\alpha L}, l^+_{\alpha R}, F^*_{\alpha L}) \quad (\alpha = 1,2,3), \tag{7.126}$$

where l_α, $\tilde{l}_\alpha$ are current-eigenstates, the eigenstates of gauge interactions. The Lagrangian is the same as (7.98) except that the chiral multiples become triplicate and their superpotential is given by

$$W = m_{\beta\alpha}\phi_{-\alpha}\phi_{+\beta}, \tag{7.127}$$

where $m_{\beta\alpha}$ may be regarded as the (β,α) elements of a mass matrix m. The mass terms of leptons l_α and sleptons $\tilde{l}_\alpha$ are easily obtained as the generalization of the mass and mass-squared terms for e and $\tilde{e}$ given in (7.102) and (7.105):

$$\mathcal{L}_m = -(\bar{l}_L m l_R + h.c.) - \{\tilde{l}^\dagger_R(m^\dagger m)\tilde{l}_R + \tilde{l}^\dagger_L(mm^\dagger)\tilde{l}_L\}, \tag{7.128}$$

where the lepton and slepton states are denoted by column vectors $l_{L,R} = (l_{1L,R}, l_{2L,R}, l_{3L,R})^t$ and $\tilde{l}_{L,R} = (\tilde{l}_{1L,R}, \tilde{l}_{2L,R}, \tilde{l}_{3L,R})^t$. It is obvious that the mass matrix for lepton and mass-squared matrices for sleptons are diagonalized simultaneously by the same unitary matrices U_L and U_R:

$$U_L^\dagger m U_R = m_{\text{diag}}, \tag{7.129}$$

$$U_R^\dagger(m^\dagger m)U_R = m^2_{\text{diag}}, \quad U_L^\dagger(mm^\dagger)U_L = m^2_{\text{diag}}, \tag{7.130}$$

and the vectors of current-eigenstates are related to the vectors of mass-eigenstates $l_{mL,R}, \tilde{l}_{mL,R}$ by $l_{L,R} = U_{L,R} l_{mL,R}$, $\tilde{l}_{L,R} = U_{L,R}\tilde{l}_{mL,R}$. On the other hand the interaction due to the neutral current coupled to the photino λ_M is given in the base of current-eigenstates as

$$\mathcal{L}_{NC} = \sqrt{2}ie\{\tilde{l}^\dagger_R\overline{\lambda_M}l_R + \tilde{l}^\dagger_L\overline{\lambda_M}l_L - h.c.\}. \tag{7.131}$$

It is clear that even if we move to the mass-eigenstates $l_{mL,R}$ and $\tilde{l}_{mL,R}$, the neutral current is still flavor-diagonal and no FCNC is induced, i.e.

$$\mathcal{L}_{NC} = \sqrt{2}ie\{\tilde{l}^{\dagger}_{mR}\overline{\lambda_M}l_{mR} + \tilde{l}^{\dagger}_{mL}\overline{\lambda_M}l_{mL} - h.c.\}, \tag{7.132}$$

since $U^{\dagger}_{L,R}U_{L,R} = I$ (I is the 3×3 unit matrix). Namely all neutral currents have no FCNC at tree level. More intuitively we may say that the superpotential (7.127) itself can be diagonalized by $U_{L,R}$ without changing the kinetic terms for chiral multiplets.

The situation changes once we introduce SUSY breaking mass-squared matrices $M^2_{L,R}$, only for sleptons (for brevity, possible mass-squared term to mix $\tilde{l}_L$ and $\tilde{l}_R$ is ignored),

$$\mathcal{L}_{SUSYbr} = -\{\tilde{l}^{\dagger}_R(M^2_R)\tilde{l}_R + \tilde{l}^{\dagger}_L(M^2_L)\tilde{l}_L\}. \tag{7.133}$$

Suppose the SUSY breaking is caused by some interactions, which are independent of flavors, such as gauge interactions. Then the SUSY breaking mass-squared matrices do not break the flavor symmetry and $M^2_{R,L}$ should be invariant under the unitary transformations by $U_{L,R}$. This argument implies that $M^2_{R,L}$ should be proportional to the unit matrix,

$$M^2_L = M^2_{SUSY,L} \cdot I, \quad M^2_R = M^2_{SUSY,R} \cdot I, \tag{7.134}$$

where $M_{SUSY,L}$ and $M_{SUSY,R}$ denote the mass scales of the SUSY breaking. As far as the SUSY breaking does not break the flavor (or horizontal) symmetry, we expect that the same argument as that in the case of exact SUSY will holds. In fact, we readily realize that the slepton mass-squared terms, including the SUSY breaking terms, are simultaneously diagonalized as the diagonalization of the lepton mass matrix,

$$U^{\dagger}_R(m^{\dagger}m + M^2_{SUSY,R}I)U_R = m^2_{\text{diag}} + M^2_{SUSY,R}I, \tag{7.135}$$

$$U^{\dagger}_L(mm^{\dagger} + M^2_{SUSY,L}I)U_R = m^2_{\text{diag}} + M^2_{SUSY,L}I. \tag{7.136}$$

Thus there does not appear FCNC in the neutral current coupled with the photino.

In the MSSM, the SUSY extension of the standard model, there appears the flavor or generation mixings in the charged currents coupled with the superpartners of $W^{\pm}$ as well. But such mixing is described by the same matrix as the KM matrix, as far as the SUSY breaking mass-squared matrices are proportional to I. We also learn from (7.135) and (7.136) that the mass-squared differences of squarks or sleptons in MSSM, which play essential roles in FCNC processes, are the same as those of quarks or leptons. For instance

$$m^2_{\tilde{c}} - m^2_{\tilde{u}} = m^2_c - m^2_u. \tag{7.137}$$

Thus the rates of FCNC processes are suppressed by the small quark mass-squared differences and/or small flavor (generation) mixings, just as in the case of the standard model. This mechanism may be called as "super-GIM" mechanism (Barbieri and Gatto, 1982; Inami and Lim, 1982).

We, however, have to be aware of the fact that a priori there is no mean to restrict the SUSY breaking mass-squared terms. In fact $M^2_{L,R}$ may be arbitrary, without contradicting with the gauge symmetry. Hence, unless there is some good reason to guarantee the flavor symmetry in the SUSY breaking masses, the model will face a serious problem of too large event rates of FCNC processes, such as $\mu \to e\gamma$ decay shown in Fig. 7.4, whose branching ratio should be extremely small, $< 4.9 \times 10^{-11}$ even if the process ever exists. In such a sense, the study of FCNC processes may be quite crucial in order to select the mechanism of SUSY breaking, which strongly characterizes the SUSY theories.

7.12 Spontaneous SUSY breaking

We have seen in the argument of SUSY QED that the masses of electron and selectron should be the same provided the SUSY holds. Experimentally, however, no selectron with the mass m_e has been observed. Thus SUSY should not be an exact symmetry in the nature and should be broken spontaneously or explicitly. Unfortunately, no successful spontaneous SUSY breaking in the sector containing ordinary quarks and leptons is known because of some phenomenological difficulty, though the spontaneous breaking is theoretically more appealing. Thus in the MSSM, for instance, explicit SUSY breaking scalar mass-squared terms, for instance, are added to the Lagrangian (by hand). We, however, discuss in this subsection the mechanisms of spontaneous SUSY breaking, since it is not only theoretically appealing, but also is expected to be the origin of the "explicit SUSY breaking". In fact, it is argued that in the MSSM embedded into a supergravity theory, for instance, the explicit SUSY breaking terms are brought into the "observed" sector of quarks and leptons via supergravity interaction from some "Higgs sector", which is decoupled from the observed sector once supergravity interaction is switched off and where the SUSY is spontaneously broken.

First let us note that the vacuum energy, i.e. the vacuum expectation value (VEV) of the Hamiltonian $E_v = \langle 0|H|0\rangle$ is the order parameter of the spontaneous SUSY breaking. This is easily seen from the relation in SUSY algebra,

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -2\ (\sigma^\mu)_{\alpha\dot{\alpha}}\ P_\mu. \tag{7.138}$$

As only σ^0 has non-zero trace, $\mathrm{Tr}\sigma^0 = -2$, we find

$$\{Q_1, \bar{Q}_{\dot{1}}\} + \{Q_2, \bar{Q}_{\dot{2}}\} = -2\mathrm{Tr}\sigma^0 P_0 = 4H. \tag{7.139}$$

As the each term in the left hand side is semi-positive definite operator, we find $E_v = \langle 0|H|0\rangle \geq 0$. If E_v vanishes we conclude SUSY is not spontaneously broken as $Q_\alpha|0\rangle = \bar{Q}_{\dot{\alpha}}|0\rangle = 0$. Thus to break SUSY spontaneously, $E_v > 0$ is necessary.

In fact if this is the case, under some SUSY transformation due to Q_α or $\bar{Q}_{\dot{\alpha}}$, the vacuum state is not invariant. Thus

$$\text{Spontaneous SUSY breaking} \quad \rightarrow \quad \langle 0|H|0\rangle > 0. \tag{7.140}$$

Actually it is quite easy to realize a SUSY model with $E_v > 0$. Let us imagine a Wess–Zumino model with a chiral superfield ϕ and a simple superpotential $W(\phi) = c\,\phi$ (c : constant). The resultant scalar potential is positive, $V = |\frac{\partial W(A)}{\partial A}|^2 = |c|^2 > 0$, and SUSY is expected to be broken, formally speaking. It, however, does not lead to any mass splitting between the scalar A and the spinor ψ components of ϕ. In fact, $W|_F = c\,F$ (F : auxiliary field), which has no self-interaction, nor mass terms for A and ψ. Thus $E_v > 0$ is necessary but not sufficient condition for "physical" SUSY breaking. (This is why the arrow in (7.140) has only one direction.) In the following we discuss possible mechanisms to get physical SUSY breaking.

In general, scalar potential is the sum of the squared absolute values of auxiliary fields,

$$V = |F_i|^2 + \frac{1}{2}(D^a)^2, \tag{7.141}$$

where the auxiliary fields are written in terms of scalar fields A_i as implied by the equations of motion,

$$F_i^* = \frac{\partial W(A)}{\partial A_i}, \quad D^a = g\; A_i^*(T^a)_i^j A_j, \tag{7.142}$$

where g is the gauge coupling and T^a are the gauge generators acting on A_i. Thus to achieve spontaneous SUSY breaking with constant scalar vacuum expectation values, necessary for Lotentz invariance, $E_v = \langle 0|V|0\rangle = |\langle F_i\rangle|^2 + \frac{1}{2}\langle D^a\rangle^2 > 0$ is needed ($\langle F_i^*\rangle = \langle 0|F_i^*|0\rangle = \frac{\partial W(<A>)}{\partial <A_i>}$, etc.). We therefore think of three possibilities to realize the spontaneous SUSY breaking:

$$\begin{aligned} &(a)\ \langle F_i\rangle \neq 0, \ \text{ irrespectively of } \langle D^a\rangle \\ &(b)\ \langle D^a\rangle \neq 0, \ \text{ irrespectively of } \langle F_i\rangle \\ &(c)\ \langle F_i\rangle = 0 \ \text{ and } \ \langle D^a\rangle = 0 \ \text{ do not hold simultaneously,} \end{aligned} \tag{7.143}$$

where in the former two cases we do not have to care whether other types of auxiliary fields develop vanishing VEV's or not. The spontaneous SUSY breaking of type (a) and (c) are well-known and were proposed by O'Raifeartaigh (O'Raifeartaigh, 1975) and Fayet–Iliopoulos (Fayet and Iliopoulos, 1974). The third type (b) (Fayet, 1976; Inami, Lim and Sakai, 1983) is less-known, but will be discussed here, as it is one logical possibility and has a nice feature that SUSY is spontaneously broken purely due to the gauge interaction.

(a) $\langle F_i\rangle \neq 0$ (O'Raifeartaigh mechanism)

We seek a model where $\langle F_i\rangle = 0$ are not satisfied for all i, i.e. at least $\langle F_i\rangle \neq 0$ for some i, no matter $\langle D^a\rangle \neq 0$ or not for some a. Hence we consider Wess–Zumino model with several chiral superfields, without considering their gauge interaction. As we have pointed out above, $\langle F \rangle \neq 0$ itself is easily realized only with 1 chiral superfield, though it does not lead to any physical SUSY breaking. O'Raifeartaigh found that to realize physically meaningful SUSY breaking at least three chiral superfields are necessary. Namely he found that a superpotential W for three chiral superfields ϕ_i $(i = 1, 2, 3)$,

$$W(\phi) = \lambda\phi_1(\phi_3^2 - M^2) + \mu\phi_2\phi_3, \tag{7.144}$$

yields the SUSY breaking, since not all of

$$\begin{aligned} F_1^* &= \frac{\partial W(A)}{\partial A_1} = \lambda(A_3^2 - M^2) = 0, \\ F_2^* &= \frac{\partial W(A)}{\partial A_2} = \mu A_3 = 0, \\ F_3^* &= \frac{\partial W(A)}{\partial A_3} = 2\lambda A_1 A_3 + \mu A_2 = 0, \end{aligned} \tag{7.145}$$

are satisfied for any VEV's of $A_{1,2,3}$. Assume, for simplicity, that M, λ, μ are all real and $\mu^2 > 2\lambda^2 M^2$, then the minimum of the scalar potential $V = \lambda^2|A_3^2 - M^2|^2 + \mu^2|A_3|^2 + |2\lambda A_1 A_3 + \mu A_2|^2$ is known to be achieved by $\langle A_1 \rangle = \langle A_2 \rangle = \langle A_3 \rangle = 0$. Thus $\langle V \rangle = \lambda^2 M^4 > 0$, as expected. Furthermore, we can check that there appear mass splitting among the masses of scalars and fermions. Since all VEV's of scalars are 0, the masses of fermions and scalars are easily read off from the quadratic terms in W and V, respectively: $\mu\psi_2\psi_3$ for the fermion mass, $-2\lambda^2 M^2 \mathrm{Re}(A_3^2) + \mu^2(|A_2|^2 + |A_3|^2)$ for the scalar mass-squared. Namely the Weyl fermions ψ_2, ψ_3 are combined to form a Dirac fermion $\Psi = (\psi_3, \bar{\psi}_2)^t$ with a mass

$$m_\Psi = \mu \tag{7.146}$$

while the scalar masses are given as

$$m_{A_1}^2 = 0,\ m_{A_2}^2 = \mu^2,\ m_{ReA_3}^2 = \mu^2 - 2\lambda^2 M^2,\ m_{ImA_3}^2 = \mu^2 + 2\lambda^2 M^2. \tag{7.147}$$

Thus the masses of fermions and scalars are not identical any more, as the consequence of SUSY breaking. We, however, also note that, even though the mass splitting does occur, the average mass-squared of fermions and scalars are still identical. To be more strict, so-called super-trace of mass-squared matrix, i.e. $\mathrm{Str}M^2 \equiv \sum_J (-1)^{2J}(2J+1)m_J^2$ vanishes when SUSY is spontaneously broken. Here m_J is the mass of the field with spin J field, where each of Majorana fermion or real scalar with mass m contributes to the sum as $-2m^2$ and m^2. For instance, in the O'Raifeartaigh model we can explicitly confirm that $\mathrm{Str}M^2 = 2 \times (0 + \mu^2) + (\mu^2 - 2\lambda^2 M^2) + (\mu^2 + 2\lambda^2 M^2) - 2 \times (2 \times \frac{1}{2} + 1)\mu^2 = 0$, where the multiplication factor 2 denotes that the corresponding masses are for complex scalars and a Dirac fermion.

It is exactly this fact of vanishing super-trace that causes the phenomenological difficulty when we attempt to construct a SUSY model where SUSY is spontaneously broken in the sector containing ordinary quarks and leptons, since it means that some of scalar partners of quarks or leptons should be lighter than corresponding quarks or leptons, which is not acceptable as we have not observed any of such superpartners.

In the O'Raifeartaigh model, once gauge interaction is included, all of the superfields $\phi_{1,2,3}$ should be singlets under gauge transformation, for W to be gauge invariant, which may be not appealing if SUSY is to be relevant for high energy theory. Thus we next consider the case (c), which realizes the spontaneous SUSY breaking in a SUSY gauge theory.

(c) $\langle F_i \rangle = 0$ and $\langle D^a \rangle = 0$ do not hold simultaneously (Fayet-Iliopoulos mechanism)

Fayet and Iliopoulos proposed a model where $\langle F_i \rangle = 0$ and $\langle D^a \rangle = 0$ are incompatible, and therefore SUSY is spontaneously broken. Thus, in this model, SUSY is broken as the result of the interplay between the superpotential, i.e the self-interaction of chiral fields, and gauge interaction, and the mechanism is different from (b), discussed below, where SUSY is broken purely due to the gauge interaction.

They essentially considered a SUSY QED discussed in section 7.8. They, however, put an additional SUSY and gauge invariant term with a real parameter ξ,

$$-\xi V|_D = -\xi D, \tag{7.148}$$

which is called Fayet–Iliopoulos D-term. (Of course, in no-Abelian gauge theory this term is no longer gauge invariant and is forbidden.)

The scalar potential for the selectrons $\tilde{e}_{R,L}$ is now modified from (7.105) into

$$V = m^2(|\tilde{e}_R|^2 + |\tilde{e}_L|^2) + \frac{1}{2}(e|\tilde{e}_R|^2 - e|\tilde{e}_L|^2 + \xi)^2. \tag{7.149}$$

We immediately realize that if $m = 0$ or $\xi = 0$, SUSY is not broken, since the vacuum states with $E_v = 0$ are realized by the VEV's $|\langle \tilde{e}_R \rangle|^2 = |\langle \tilde{e}_L \rangle|^2 - \frac{\xi}{e}$ or $|\langle \tilde{e}_R \rangle| = |\langle \tilde{e}_L \rangle| = 0$, respectively. On the other hand if both of m and ξ are non-vanishing, SUSY is spontaneously broken, since the two conditions

$$|\tilde{e}_R|^2 + |\tilde{e}_L|^2 = 0, \quad e|\tilde{e}_R|^2 - e|\tilde{e}_L|^2 + \xi = 0, \tag{7.150}$$

are not satisfied simultaneously.

Depending on the sign of $m^2 - e\xi$ (assuming $\xi > 0$), the minimum of the potential V is given by the following VEV's

$$\begin{aligned} \langle \tilde{e}_R \rangle &= \langle \tilde{e}_L \rangle = 0, \quad \text{for } m^2 - e\xi > 0, \\ \langle \tilde{e}_R \rangle &= 0, \ \langle \tilde{e}_L \rangle = \frac{\sqrt{e\xi - m^2}}{e}, \quad \text{for } m^2 - e\xi < 0. \end{aligned} \tag{7.151}$$

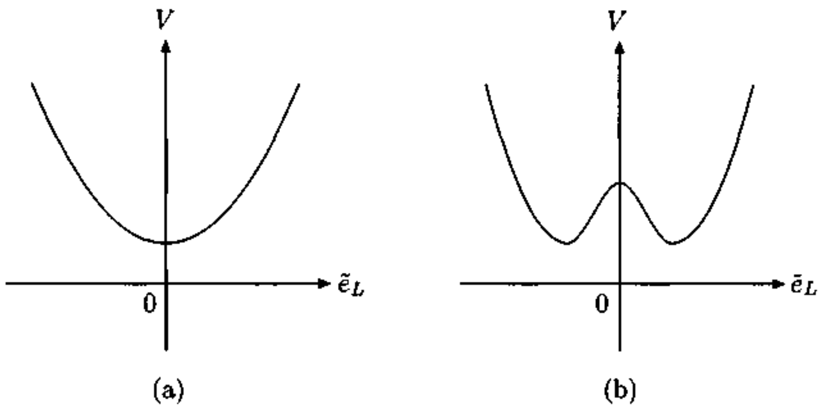


Fig. 7.5

In the former case, though SUSY is broken, the $U(1)$ gauge symmetry is not spontaneously broken with vanishing VEV's, while in the latter case both symmetries are spontaneously broken (see Fig. 7.5).

In the case $m^2 - e\xi > 0$, for instance, the masses of electron and selectrons are given as

$$m_e = m, \ m^2_{\tilde{e}_R} = m^2 + e\xi, \ m^2_{\tilde{e}_L} = m^2 - e\xi. \tag{7.152}$$

Thus, again we get the remarkable relation

$$\mathrm{Str}M^2 = 2 \times (m^2 + e\xi) + 2 \times (m^2 - e\xi) - 2 \times (2 \times \frac{1}{2} + 1)m^2 = 0. \tag{7.153}$$

(b) $\langle D^a \rangle \neq 0$ (The third possibility)

As the third possibility we consider a case where $\langle D^a \rangle = 0$ are not simultaneously satisfied by any choice of VEV's of scalar fields, leading to $E_v > 0$. Namely, in this case SUSY is spontaneously broken purely due to gauge interaction. For the transparency of the argument, here we neglect superpotential, since the presence is irrelevant for the mechanism to work.

Ignoring the superpotential, the scalar potential, the values of VEV's and the breaking of SUSY is completely determined by the choice of gauge group and the group representations of matter chiral superfields. One restriction is that the gauge group should have at least one $U(1)$ factor, since otherwise Fayet–Iliopoulos D-term is not allowed and supersymmetric vacuum state is easily realized for all vanishing VEV's of scalar fields.

The simplest model to realize the SUSY breaking turns out to be $SU(2) \times U(1)$ SUSY gauge theory (Fayet, 1976; Inami, Lim and Sakai, 1983) with Fayet–Iliopoulos D-term $-\tilde{g}\xi D$ (we denote gauge couplings of $SU(2)$ and $U(1)$ by g and $\tilde{g}$) and two chiral superfields

$$\phi \ \ (Q = \frac{1}{2}), \quad \phi' \ \ (Q = -\frac{1}{2}), \quad (Q : U(1) \text{ charge}), \tag{7.154}$$

both belonging to the doublet representation of $SU(2)$. Let us note the assignment of the $U(1)$ charges $Q = 1/2, \ -1/2$ is such that the triangle gauge anomaly disappears. By use of the equations of motion for D auxiliary fields, the auxiliary fields are written in terms of scalar fields, which we denote by the same letters ϕ and ϕ', as

$$\begin{aligned} D^a &= g(\phi^\dagger T^a \phi + \phi'^\dagger T^a \phi') \ \ (T^a = \frac{\sigma_a}{2}) \ \text{ for } SU(2) \\ D &= \tilde{g}(\frac{1}{2}\phi^\dagger \phi - \frac{1}{2}\phi'^\dagger \phi' + \xi) \ \text{ for } U(1). \end{aligned} \tag{7.155}$$

By use of the $SU(2) \times U(1)$ symmetry of the theory, we can assume the following form of the VEV's of scalar fields, without loss of generality,

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ x \end{pmatrix}, \qquad \langle \phi' \rangle = \begin{pmatrix} ye^{i\theta} \\ z \end{pmatrix}, \tag{7.156}$$

where x, y, z and θ are all real.

Then the conditions $\langle D^a \rangle = 0$ and $\langle D \rangle = 0$ read

$$\begin{aligned} yz &= 0 \quad \text{for } \langle D^1 \rangle = \langle D^2 \rangle = 0, \\ x^2 - y^2 + z^2 &= 0 \quad \text{for } \langle D^3 \rangle = 0, \\ x^2 - y^2 - z^2 + 2\xi &= 0 \quad \text{for } \langle D \rangle = 0. \end{aligned} \tag{7.157}$$

It is easy to check that for $\xi \neq 0$ these three conditions are not met simultaneously By any choice of x, y and z. Thus, we conclude that $E_v = \langle V \rangle = \frac{1}{2}\langle D^a \rangle^2 + \frac{1}{2}\langle D \rangle^2 > 0$ and SUSY is spontaneously broken, irrespectively of the choice of the superpotential.

This argument can be generalized to $SU(n) \times U(1)$ SUSY gauge theories. It turns out that SUSY is spontaneously broken provided the number of pairs of chiral superfields belonging to n and $\bar{n}$ representations of $SU(n)$ with opposite $U(1)$ charges is less than n (Inami, Lim and Sakai, 1983). The $SU(2) \times U(1)$ theory with a single pair of chiral superfields just discussed above is the simplest example.

Problems

7.1 Prove the following relations concerning the 2-component spinors (anti-commuting Grassman numbers) θ and $\bar{\theta}$ and the 2×2 matrices σ^μ and $\bar{\sigma}^\mu$. In $\theta\theta$, $\theta\sigma\bar{\theta}$ etc., spinor indices have been suppressed:

$$a. \qquad \theta^\alpha \theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta, \ \theta_\alpha \theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta,$$

$$\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta},\ \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \tag{7.158}$$

b. $$(\theta\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta}) = \frac{1}{2}g^{\mu\nu}(\theta\theta)(\overline{\theta\theta}) \tag{7.159}$$

c. $$(\theta\psi_1)(\theta\psi_2) = -\frac{1}{2}(\psi_1\psi_2)(\theta\theta) \tag{7.160}$$

d. $$\det(\sigma^\mu p_\mu) = \det(\bar{\sigma}^\mu p_\mu) = p_\mu p^\mu \tag{7.161}$$

e. $$(\overline{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^\mu)_{\beta\dot{\beta}} \tag{7.162}$$

f. $$\{\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu\}_\alpha{}^\beta = 2g^{\mu\nu}\delta_\alpha{}^\beta \tag{7.163}$$

g. $$\{\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu\}^{\dot{\alpha}}{}_{\dot{\beta}} = 2g^{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \tag{7.164}$$

h. $$\mathrm{Tr}(\sigma^\mu\overline{\sigma}^\nu) = 2g^{\mu\nu} \tag{7.165}$$

i. $$(\sigma^\mu)_{\alpha\dot{\alpha}}(\overline{\sigma}_\mu)^{\dot{\beta}\beta} = 2\delta_\alpha{}^\beta\delta_{\dot{\alpha}}{}^{\dot{\beta}} \tag{7.166}$$

j. $$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\lambda - \overline{\sigma}^\lambda\sigma^\nu\overline{\sigma}^\mu = -2i\epsilon^{\mu\nu\lambda\kappa}\overline{\sigma}_\kappa\ (\epsilon_{0123} = -\epsilon^{0123} = 1) \tag{7.167}$$

k. $$\sigma^\mu\overline{\sigma}^\nu\sigma^\lambda - \sigma^\lambda\overline{\sigma}^\nu\sigma^\mu = 2i\epsilon^{\mu\nu\lambda\kappa}\sigma_\kappa \tag{7.168}$$

l. $$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\lambda + \overline{\sigma}^\lambda\sigma^\nu\overline{\sigma}^\mu = -2(g^{\mu\lambda}\overline{\sigma}^\nu - g^{\nu\lambda}\overline{\sigma}^\mu - g^{\mu\nu}\overline{\sigma}^\lambda), \tag{7.169}$$

m. $$\sigma^\mu\overline{\sigma}^\nu\sigma^\lambda + \sigma^\lambda\overline{\sigma}^\nu\sigma^\mu = -2(g^{\mu\lambda}\sigma^\nu - g^{\nu\lambda}\sigma^\mu - g^{\mu\nu}\sigma^\lambda). \tag{7.170}$$

7.2 Verify the relation in (7.31),

$$G(0,\epsilon,\overline{\epsilon})\cdot G(x^\mu,\theta,\overline{\theta}) = G(x^\mu + i\theta\sigma^\mu\overline{\epsilon} - i\epsilon\sigma^\mu\overline{\theta}, \theta+\epsilon, \overline{\theta}+\overline{\epsilon}). \tag{7.171}$$

7.3 Find the propagator of the auxiliary field F of a chiral superfield (consulting, for instance, with the Lagrangian given in (7.61)). Apply the propagator to the calculation of the diagram in the left of Fig. 7.2(a), and show that the coupling $-4ig^2$ should be assigned for the vertex of the diagram in the right of Fig. 7.2(a).

7.4 Show that in the simplified Wess–Zumino model with only one chiral superfield ϕ, whose Lagrangian is given in (7.66), there is an exact cancellation of the quadratic divergence between the diagrams Fig. 7.2(a) and (b). Show also that the another type of diagrams with the loop of fermionic field, coming form the contraction of two identical vertices, e.g. $A\bar{\psi}_M\frac{1+\gamma_5}{2}\psi_M$, does not yield the quadratic divergence, even for $m \neq 0$.

7.5 Verify (7.87) and (7.88).

Chapter 8

PRECISION TEST OF ELECTROWEAK RADIATIVE CORRECTIONS AND NEW PHYSICS

8.1 The Meaning of Precision Test of Electroweak Radiative Corrections

The standard model of elementary particles, especially its electroweak sector, was devised as a renormalizable gauge theory with massive vector bosons such as Z^0 and $W^{\pm}$. Already at the classical (tree) level, without quantum corrections, the standard model has its characteristic properties and such properties have been extensively tested experimentally. We may pick up some typical examples of such tests: discovery of the predicted neutral current processes, confirmation of the existence of Z^0 and $W^{\pm}$ with their predicted masses, forward–backward asymmetries A_{FB} in the scattering $e^+e^- \to f\bar{f}$ (f : μ, etc.).

The genuine features of the theory as the renormalizable theory, however, should be studied by comparing the calculated (finite but small) quantum effects of the theory on the physical observables, i.e. radiative corrections, with the data obtained in the precision experiments, such as LEP, CDF, etc.

Let us recall how the renormalization procedure goes, namely how the finite (calculable) radiative corrections are obtainable in the renormalizable theory. To avoid unnecessary complication, let us focus on the gauge sector, namely the part of the Lagrangian, which contains gauge bosons. Then we have 3 bare parameters: 2 gauge couplings g, g' of $SU(2)_L$ and $U(1)_Y$ and the vacuum expectation value (VEV) of the neutral Higgs boson, v. What we should do first is to calculate observables, which have been excellently well-measured experimentally, i.e.

$$\begin{aligned} M_Z &= 91.150(30) \text{ GeV (from LEP, SLC)}, && (8.1)\\ G_F &= 1.16637(2) \times 10^{-5} \ (\text{GeV})^{-2} \ (\text{from } \mu \to e\nu_\mu\bar{\nu}_e), && (8.2)\\ \alpha &= 137.0359895(61)^{-1} \ (\text{from } g-2 \text{ of electron}), && (8.3) \end{aligned}$$

including quantum effects in terms of the 3 bare parameters. Namely, we calculate these quantities as the functions of 3 bare parameters and other parameters, such as m_t, m_H:

$$M_Z(g,g',v;m_t,m_H,\cdots),\ G_F(g,g',v;m_t,m_H,\cdots),\ \alpha(g,g',v;m_t,m_H,\cdots), \quad (8.4)$$

which at the first sight have UV (ultra-violet) divergences, but actually are finite "renormalized" quantities. (Or we may say that the bare parameters themselves are UV divergent.) Equating these 3 functions with the measured values listed above provides formulae, which enables us to express the bare parameters in terms of observables M_Z etc.: $g = \phi(M_z, G_F, \alpha; m_t, m_H, \cdots)$, etc. Then, we may calculate any other observables in terms of bare parameters, $f(g, g', v; m_t, m_H, \cdots)$, which are superficially divergent. Once the functions $\phi(M_Z, \cdots)$ are substituted for the bare parameters g etc., they become the functions of well-measured quantities: $f(g, g', v; m_t, m_H, \cdots) = \hat{f}(M_z, G_F, \alpha; m_t, m_H, \cdots)$. The obtained quantities $\hat{f}$ should be all finite, as long as the theory is renormalizable. These are real predictions of the renormalizable theory.

The above argument on the quantum effects is easily understood, relying on (local gauge invariant) operators. As is well-known in 4 space-time dimension, a renormalizable theory should contain only marginal or relevant operators with mass dimension $d = 4$ or $d < 4$ to start with. After including quantum effects, there also appear irrelevant operators with $d > 4$, together with the marginal and relevant operators. The marginal and relevant operators get ultra-violet(UV) divergent corrections on the coefficients of their operators, such as Λ^2 or $\ln\left(\frac{\Lambda}{\mu}\right)$, with Λ and μ being momentum cutoff and the renormalization scale. (The momentum cutoff potentially break local gauge invariance and usually alternative regularization methods such as dimensional regularization may be used.) The quantum corrections to g and g' and the corrections to gauge boson mass-squared M_Z^2, M_W^2, keeping the relation of bare masses,

$$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1, \quad (8.5)$$

belong to this category. On the other hand the coefficients of irrelevant operators, induced by the quantum effects, should be automatically finite; otherwise the UV-divergences cannot be removed, since there are no irrelevant operators nor counterterms in the original Lagrangian. The physical observable $\Delta\rho \equiv \rho - 1$ or equivalently $M_Z^2\cos^2\theta - M_W^2$, caused, e.g., by a mass splitting between the members of $SU(2)_L$ doublet, is a typical example. Apparently, it seems that the radiatively induced $\Delta\rho$ is UV-divergent, as the gauge boson mass-squared seems to be the coefficient of a $d = 2$ relevant operator. This apparent contradiction is resolved once we invoke to the language of (local) gauge invariant operators with Higgs field included. As far as the theory to start with is gauge invariant, every operators induced or corrected by quantum or radiative effects should be all gauge invariant. This is, say, a picture in the "symmetric phase" of the theory. Of course the gauge boson mass-squared stems from the spontaneous symmetry breaking (SSB), i.e. in the "broken phase" of the theory. Thus the mass-squared (difference) relevant for $\Delta\rho$ should be understood as the result of the replacement of the (neutral) Higgs field by its VEV in an irrelevant operator composed of Higgs field, as well as gauge bosons. In fact we will see in the later section that $\Delta\rho$ can be regarded as the coefficient of a $d = 6$ operator. This is why the radiatively induced $\Delta\rho$ is finite.

The precision tests of such finite radiative "corrections" (quantum effects) to electroweak parameters, such as gauge couplings and gauge boson masses, are quite important to check the validity of the standard model, since these finite effects are genuine predictions of the model. For instance such tests are performed by the precision measurements of various parity violating asymmetries such as forward-backward asymmetry or LR asymmetry in e^+e^- scattering (LEP etc.). The heavy particles of the SM, like t quark and Higgs, and still unknown heavy particles predicted by theories beyond the standard model ("New Physics") do not directly appear in the external lines of Feynman diagrams of lower energy processes. They appear only or mainly through their contributions to gauge boson self-energies, i.e. through indirect "oblique corrections". Strictly speaking, e.g. the t quark may contribute to $\bar{b}bZ$ vertex, and these extra contributions have to be estimated depending on each case. It, however, is still true that the oblique corrections exist universally when we consider the radiative corrections of heavy particles and are worth general investigations.

As the coefficients of irrelevant operators have negative mass dimensions ($d < 0$), we may expect that the finite quantum effects due to heavy particles may be suppressed by the inverse powers of the heavy particle masses ("decoupling" of heavy particles in low energy processes). But it turns out that this is not always true, and some radiative corrections are not suppressed by the inverse powers. Such "non-decoupling" effects in oblique corrections potentially appear only in restricted number of "oblique parameters", S, T and U, which parameterize electro-weak radiative corrections. The precision tests of the 3 parameters S, T, U are quite important not only to get useful information of the heavy particles in the standard model, m_t and m_H, but also to test various theories of New Physics. In fact, (original version of) "technicolor" theory was ruled out as the result of the precision test of the S-parameter (Peskin and Takeuchi, 1990).

The purpose of this chapter is to discuss the precision test of the radiative corrections to the electro-weak parameters in some detail, not only due to the heavy (comparable or greater than M_W) t quark or Higgs in the standard model, but also due to the heavy particles predicted by various New Physics. We, however, assume that the gauge group we take is the same as the standard model, i.e. $SU(2)_L \times U(1)_Y$.

8.2 Decoupling and Non-decoupling

As we discussed above, precision test of the radiative corrections due to heavy particles provides useful information on the properties of heavy particles such as their masses, or even rules out their existence. The prediction of c quark mass by Gaillard and Lee (Gaillard and Lee, 1974) before its discovery, utilizing the data on the flavor changing neutral current processes of neutral kaons K^0, $\bar{K}^0$, is the remarkable example.

Heavy particles are heavy in the sense that the processes we treat are those for lighter particles at lower energies. That is why to get information on their masses indirectly through radiative corrections is meaningful. Heavy particles, even though they cannot appear as the real states in the lower energy processes, may participate in the processes as the virtual or intermediate states. We, however, naively anticipate that the effects are suppressed by the inverse powers of the heavy particle masses, which we generically denote by M, since the propagators of heavy particles behave as (for scalars)

$$\frac{i}{k^2 - M^2} \simeq -\frac{i}{M^2}, \tag{8.6}$$

in lower ($|k^2| \ll M^2$) energies. At the tree level, this replacement is justified. In the loop diagrams, however, the replacement is no longer justified, since the 4-momentum k_μ varies as the loop momentum in the integration. Thus whether the heavy particle contributions are really suppressed by the inverse powers of their masses or not is a non-trivial question to be addressed.

In gauge theories without SSB, having only parity preserving vector-like (non-chiral) interactions, such as QED or QCD, so-called "decoupling theorem" holds (Appelquist and Carrazone, 1975): the contributions of heavy particles with mass M to physical observables are suppressed by the powers of $1/M$. Let us take the case of QED as an example. Suppose we have a heavy charged lepton E^- with the mass M, say a partner of ordinary electron belonging to hypothetical heavy generation. Let us consider its contributions as virtual states in the processes described by Feynman diagrams with n external photon lines (see Fig. 8.1).

The quantum effects provide effective Lagrangian for external photon fields

$$\mathcal{L}_{eff} = \sum_i c_i(M)\, O_i, \tag{8.7}$$

where O_i denote local gauge invariant operators with n photon fields A_μ and c_i are their coefficient functions, which depend M in addition to $\alpha = \frac{e^2}{4\pi}$ as the result of loop integration of internal lines. Note that the mass dimension of the operators O_i are greater than n, $d_i > n$, in general. This is because to make gauge invariant operators, we need field strength $F_{\mu\nu}$ rather than A_μ itself, which has derivatives of A_μ, thus making the dimension of the operators higher than n. Thus, as is clear from dimensional analysis, for $n > 4$ the coefficient functions are suppressed by

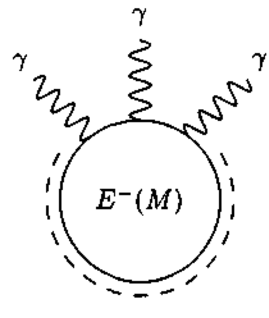


Fig. 8.1

inverse powers of M, $1/M^{d_i-4}$ ($d_i \geq n > 4$). What all of these are saying is that the coefficients of irrelevant operators are suppressed by the inverse powers of M.

This kind of dimensional analysis no longer holds for the case of marginal operator, which appears in the case of $n = 2$, i.e. self-energy diagram. This just corresponds to the correction to $F_{\mu\nu}F^{\mu\nu}$. (We also get operators with higher derivatives, which we ignore as they are contained in the category of irrelevant operators.) The coefficient is dimensionless and behave as $\int_0^1 dt \ln\left(\frac{\Lambda^2}{M^2-t(1-t)q^2}\right)$, with Λ, t being momentum cutoff and Feynman parameter. Obviously, this coefficient is not suppressed by large M. This correction, however, is not a prediction of the theory as it is UV-divergent. After imposing a renormilization condition that the correction disappear at $-q^2 = \mu^2$, which is met by adding a counterterm made from bare parameters, the correction is rendered to be finite: $\int_0^1 dt \ln\left(\frac{\Lambda^2}{M^2-t(1-t)q^2}\right) - (-q^2 \to \mu^2)$, which may be approximated at low energies $\mu^2, |q^2| \ll M^2$ to be, $-\int_0^1 dt\, t(1-t)\, \frac{q^2+\mu^2}{M^2} = -\frac{1}{6}\frac{q^2+\mu^2}{M^2} \ll 1$. Hence, again the net effect of the heavy particle is suppressed by an inverse power of M, though it does contribute to the renormalization of the bare parameters. These are contents of the "decoupling theorem".

This situation is psychologically good, since when we calculate some physical quantities just relying on the standard model we do not have to worry about what really is the theory in higher energies where unknown heavy particles are supposed to exist: these heavy particles may give some extra contributions but they are safely small. However, it also says that we cannot have significant information concerning the properties, such as masses, of the heavy particles or concerning New Physics. Fortunately, in chiral theories with SSB like the standard model, some "non-decoupling" effects are known to exist. As the above argument of decoupling theorem is a convincing one, one may wonder what is new in such theories. The key ingredient, which leads to the non-decoupling, is the fact that now (at least some of) the coupling constants are proportional to masses of the relevant particles. More precisely, for instance in the standard model, all massive particles (including Higgs itself) get their masses only through spontaneous gauge symmetry

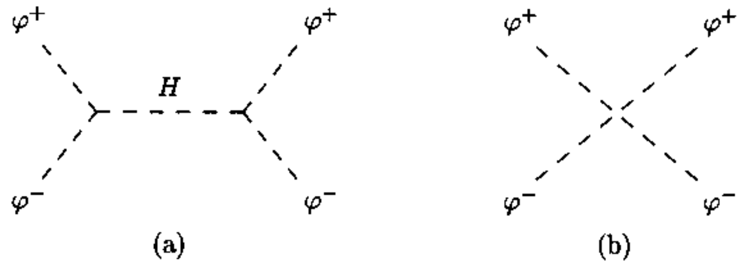


Fig. 8.2

breaking (SSB), i.e. through their couplings with the Higgs field, which develops VEV. Thus when some particle is heavy that means its coupling with the Higgs, such as Yukawa couplings and Higgs self-coupling, are strong. (Hereafter we will assume that the couplings are not extremely strong, not to spoil the perturbative expansion.) For instance the Yukawa coupling of the top quark f_t is proportional to m_t, $f_t = m_t/(v/\sqrt{2}) = \frac{g}{\sqrt{2}}\frac{m_t}{M_W}$. Such couplings appear in the vertices of Feynman diagrams and therefore in the numerators of transition amplitudes of the processes we consider, and the above argument on the decoupling theorem, solely relying on dimensional analysis, is no longer valid. We thus anticipate that in some suitably chosen processes, heavy particle contributions may be enhanced by (positive) powers of M.

We will list up some known examples of such non-decoupling effects of heavy particles in the following.

(a) Non-linear sigma model

The non-decoupling effect is obtainable already at tree level. This seems to contradict with our intuition that the propagator of heavy particles are suppressed by $1/M^2$ as we have discussed above. Concerning the Higgs self-interaction in the standard model, however, this naive argument no longer holds. Let us consider 4-point scattering amplitude of charged Higgs φ^+ ("would-be Goldstone" particle) as shown in Fig. 8.2:

In Fig. 8.2(a), the exchange of the "physical" neutral Higgs H yields a propagator $i/(q^2 - M_H^2) \simeq -i(1/M_H^2) - i(q^2/M_H^4)$, which is suppressed by $1/M_H^2$. On the other hand the vertices are both enhanced by an factor proportional to M_H^2, i.e. by $-ig\frac{M_H^2}{2M_W}$. Thus this Higgs-exchange diagram provides an effective Lagrangian (in momentum space)

$$-i\,\frac{1}{2}\frac{i}{q^2 - M_H^2}\,(-ig\frac{M_H^2}{2M_W^2})^2\,(\varphi^+\varphi^-)^2 \simeq \frac{g^2}{8}\,(\frac{1}{M_H^2} + \frac{q^2}{M_H^4})\,\frac{M_H^4}{M_W^2}\,(\varphi^+\varphi^-)^2. \quad (8.8)$$

The first term without q^2 implies we have effectively got a contact interaction enhanced as M_H^2. Actually this term is exactly cancelled by the original quartic interaction, shown in the second diagram Fig. 8.2(b), $-\frac{g^2 M_H^2}{8M_W^2}(\varphi^+\varphi^-)^2$. Thus in the large M_H or equivalently low energy ($|q^2| \ll M_H^2$) limit, we have only derivative interaction $-\frac{g^2}{8M_W^2}(\varphi^+\varphi^-)\,\Box(\varphi^+\varphi^-)$. This term is independent of M_H^2 and indicates a non-decoupling effect of heavy Higgs. We may attach the lines of φ^+, φ^- to the internal line of H as many times as we want to get higher dimensional operators of the fields φ. Such operators turn out to be all derivative interactions and to be summarized in a compact form ($v/\sqrt{2}$ being the VEV of the Higgs)

$$\frac{v^2}{4}\,\mathrm{Tr}\,(\partial_\mu U\,\partial^\mu U),\quad U \equiv e^{i\frac{G^i\tau^i}{v}},\quad (G^+ = -\frac{G^1+iG^2}{\sqrt{2}} = -i\varphi^+, G^3 = \sqrt{2}\mathrm{Im}\varphi^0). \tag{8.9}$$

This Lagrangian contain an infinite number of non-linear terms, and corresponds to a description of the Higgs doublet ϕ in terms of "polar coordinates system" instead of ordinary Cartesian coordinates:

$$\Phi \equiv (\tilde{\phi}\phi) \equiv \frac{v+H}{\sqrt{2}}\,U. \tag{8.10}$$

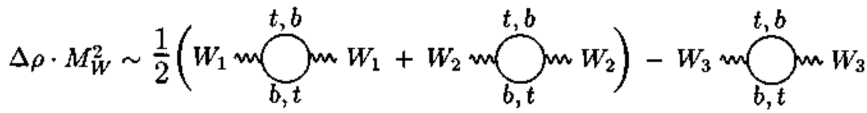
This non-linear realization of the Higgs field is the counterpart of non-linear sigma model in QCD, where φ^i should be understood as 3 pion fields π^i ($i = 1,2,3$), which are only particles appearing in the (very) low energy regime of QCD. It is now clear why the contact interaction term disappeared: in the polar coordinate system the Higgs potential is independent of φ^i, as $U^\dagger U = I$ (I : unit matrix).

(b) $\Delta\rho$

We have defined the ρ-parameter as the ratio of charged gauge boson mass-squared to that of the neutral gauge boson (with an extra factor $\cos^2\theta_W$), which is exactly 1 at the tree or classical level. The parameter, however, deviates from 1, i.e. $\Delta\rho \equiv \rho - 1 \neq 0$, once the quantum correction is taken into account. Let us consider the quantum correction due to the $SU(2)_L$ doublet $(t,\ b)^t$. If there ever exists extra generation of quarks and leptons, the doublet may be replaced by the doublet of fourth generation quarks $(t',\ b')^t$ as well. Since $W_3^\mu = \cos\theta_W\ Z^\mu + \sin\theta_W\gamma^\mu$ and the photon field γ^μ never gets mass correction due to the unbroken $U(1)_{em}$ symmetry, the quantum correction to $\cos^2\theta_W\ M_Z^2$ is equivalent to the correction to the operator $W_{3\mu}W_3^\mu$, with 3 denoting the third component of $SU(2)$ adjoint representation. So at 1-loop level, namely as far as $O(\alpha)$ corrections (denoted by the quantities accompanied by δ) are concerned,

$$\Delta\rho = \delta(\frac{M_{W^+}^2}{M_{W_3}^2}) = \frac{M_W^2 + \delta M_{W^+}^2}{M_W^2 + \delta M_{W_3}^2} - 1 \simeq \frac{1}{M_W^2}(\frac{\delta M_{W_1}^2 + \delta M_{W_2}^2}{2} - \delta M_{W_3}^2), \tag{8.11}$$

where we have replaced the mass of the charged gauge boson by $M_{W_{1,2}}$: $M_{W^+}^2 W_\mu^+ W^{-\mu} = M_{W^+}^2 \frac{W_{1\mu}W_1^\mu + W_{2\mu}W_2^\mu}{2}$. (8.11) is graphically expressed in Fig. 8.3, where $\Pi_{ab}(0)$ ($a,b = 1-3$) denote the scalar part of the vacuum polarization tensors



$$\sim \frac{1}{2} \Big(\Pi_{11}(0) + \Pi_{22}(0) \Big) - \Pi_{33}(0)$$

Fig. 8.3

of $W_{a\mu}W_{b\nu}$ 2 point functions (after factoring out $g_{\mu\nu}$) at $q^2 = 0$. Let us note that the scalar parts at 0 momentum correspond to the quantum corrections to gauge boson mass-squared.

The contributions of the $(t,\ b)^t$ doublet is readily obtained by modifying the original calculation by Veltman (Veltman, 1977) for the contribution of heavy lepton doublet to be

$$\Delta\rho = \frac{3\alpha}{16\pi \sin^2\theta_W} \frac{1}{M_W^2} (m_t^2 + m_b^2 - \frac{2m_t^2 m_b^2}{m_t^2 - m_b^2} \ln \frac{m_t^2}{m_b^2}), \tag{8.12}$$

where the factor 3 comes from the colors of quarks. One may check that this expression vanishes in the limit of "degenerate doublet" $m_t = m_b$. (The factor $\frac{1}{m_t^2 - m_b^2} \ln \frac{m_t^2}{m_b^2}$ is not singular in this limit, but just yields a derivative of the logarithmic function, $1/m_t^2$. If we take the limit of $m_b \ll m_t$, as is realized in the nature, we get

$$\Delta\rho \simeq \frac{3\alpha}{16\pi \sin^2\theta_W} \frac{m_t^2}{M_W^2}, \tag{8.13}$$

which clearly shows the enhancement of the quantum effect of heavy top quark, proportional to m_t^2, i.e. a non-decoupling phenomenon.

(c) Flavor Changing Neutral Current (FCNC) Processes

As we have already mentioned, Gaillard and Lee could predict the c quark mass before its discovery (Gaillard and Lee, 1974), studying the quantum effects of the c quark in low energy FCNC processes of neutral kaons K^0, $\bar{K}^0$, such as $K^0 \leftrightarrow \bar{K}^0$ mixing. As the neutral kaons are the bound states of d and s quarks, $K^0 \sim \bar{s}\gamma_5 d$, $\bar{K}^0 \sim \bar{d}\gamma_5 s$, this mixing is caused by an elementary process $\bar{s}d \leftrightarrow s\bar{d}$. In this transition, flavor quantum number "strangeness" S changes by two units, $|\Delta S| = 2$, while there is no change of the electric charge. This is why such processes are called Flavor Changing Neutral Current (FCNC) processes. We will discuss these FCNC processes in some detail in the next chapter. As we discuss there, since there is no FCNC interaction at the tree level of the standard model, the $\bar{s}d \leftrightarrow s\bar{d}$ transition is induced by the loop diagram, called "box diagram" (Fig. 8.4).

When Gaillard and Lee discussed the process the third generation had not been confirmed. But now its existence is known and it is an important question to ask if the heavy t quark gives a non-decoupling quantum effect to this process, thus making

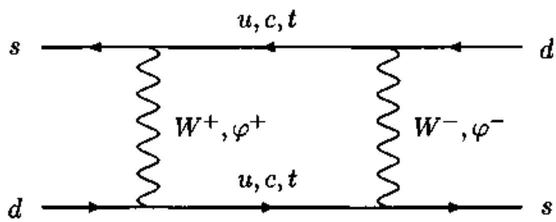


Fig. 8.4

it meaningful to estimate m_t or its flavor mixings with lower generation quarks by studying this $K^0 \leftrightarrow \bar{K}^0$ mixing, just following the philosophy of Gaillard and Lee. For such purpose it is necessary, since $m_t > M_W$, to calculate the box diagram exactly without assuming the internal quark mass is much smaller than M_W, which was adopted by Gaillard-Lee's calculation in the view of the fact $m_c \ll M_W$. The result of such exact calculation (Inami and Lim, 1981) is written in the form of 4-Fermi effective Lagrangian (see (9.11))

$$\mathcal{L}_{eff}^{|\Delta S|=2} = \frac{\alpha G_F}{4\sqrt{2}\pi \sin^2\theta_W} \sum_{i,j=c,t} (V_{is}^* V_{id})(V_{js}^* V_{jd})\ E(x_i, x_j)\ (\bar{s}\gamma_\mu L d)(\bar{s}\gamma^\mu L d), \quad (8.14)$$

where $L = \frac{1-\gamma_5}{2}$, V_{is} etc. are KM matrix elements and the coefficient function $E(x_i, x_j)$ denotes the contributions of internal up-type quarks with masses m_i, m_j ($x_i \equiv \frac{m_i^2}{M_W^2}$). The (pure) t quark contribution reads as,

$$E(x_t) \equiv E(x_t, x_t) = -\frac{3}{2}\left(\frac{x_t}{x_t - 1}\right)^3 \ln x_t - \left[\frac{1}{4} - \frac{9}{4}\frac{1}{x_t - 1} - \frac{3}{2}\frac{1}{(1 - x_t)^2}\right] x_t \quad (8.15)$$

For the imaginative limit of $x_t \gg 1$, i.e. $m_t^2 \gg M_W^2$ (though it is not so bad approximation), the function behaves as

$$E(x_t) \simeq -\frac{x_t}{4} = -\frac{1}{4}\frac{m_t^2}{M_W^2}, \quad (8.16)$$

which behaves as m_t^2 and is another example of the non-decoupling effect of the heavy fermion. It is worth noticing that for c quark, with $x_c \ll 1$, $E(x_c) \simeq -x_c$, which just recovers the result of Gaillard–Lee. The calculation (Inami and Lim, 1981) also shows that similar enhancement proportional to m_t^2 appears in the induced FCNC vertex $\bar{s}dZ$ ("Z-penguin"), but not in $\bar{s}d\gamma$ vertex (ordinary "penguin"), where we have only logarithmic enhancement $\ln \frac{m_t}{m_{c,u}}$ due to the property of the CVC. As far as we know the heavy fermion contributions to $\Delta\rho$ and these FCNC processes are only known examples where we have an quadratically growing non-decoupling effects. Interestingly, it turns out that (at least at 1-loop level) the contribution of a heavy Higgs to $\Delta\rho$ does not grow as M_H^2, but grows only as $\ln \frac{M_H}{M_W}$.

We have seen that the interesting non-decoupling effects arise in chiral theories with SSB. Even in such chiral theories, however, heavy particles effects may be decoupling. It is the case when heavy particle mass is not provided by SSB, i.e. not through large couplings with Higgs, but by a new large mass scale, independent of M_W, characterizing some New Physics. Such new mass scales are inevitably invariant concerning the gauge symmetry of the standard model, since otherwise it will break the gauge symmetry at much higher energies than M_W, which contradicts with reality. In such a sense, the situation is similar to the case of the decoupling phenomena we discussed above about a heavy lepton E^-, whose large mass was also gauge invariant and came to the denominator of the coefficients of the effective operators. We will list up some typical examples of such decoupling effects.

(d) Seesaw mechanism

In Chapter 6, we discussed the seesaw mechanism. The seesaw mechanism is known to be a scenario which naturally lead to light Majorana neutrinos. As we have already seen in 6.2, the small Majorana mass $\frac{m_D^2}{m_R}$, with m_D, m_R being a Dirac mass and the Majorana mass for ν_R, is induced through a tree level diagram in Fig. 6.1. The fact that the mass is suppressed by $1/m_R$ can be understood as to indicate the decoupling effect of the intermediate state ν_R. Let us note that $m_R (\gg M_W)$ is $SU(2)_L \times U(1)_Y$ invariant mass, which may be related with the new mass scale of, say, left-right symmetric model of electro-weak interaction $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ (Mohapatra and Senjanovic, 1980) (B, L denote baryon and lepton numbers), where $SU(2)_R$ symmetry and therefore parity is spontaneously broken at the scale $\mathcal{O}(m_R)$.

(e) The contribution of super-partners to $\Delta\rho$

As was discussed in Chapter 7, supersymmetric theories predict the presence of super-partners of ordinary particles. Here let us consider the contribution of the $SU(2)_L$ doublet $(\tilde{u}, \tilde{d})^t$, the superpartners of light generation quarks $(u, d)^t$. As the partners have not been observed in, e.g. e^+e^- collider experiments, these partners should be heavy: $m_{\tilde{u}}^2 = m_u^2 + M_{SUSY}^2$, $m_{\tilde{d}}^2 = m_d^2 + M_{SUSY}^2$, where M_{SUSY} is "SUSY breaking mass scale", denoting the extent in which SUSY is broken. (We have ignored the effect of possible mixing between the super-partners of left- and right-handed quarks etc., to avoid unnecessary complication.) These partners are heavy not because the Yukawa couplings are large ($m_{u,d} \ll M_W$), but because the gauge invariant M_{SUSY} is large. We thus anticipate the decoupling of these partners in their quantum effects. Their contribution to $\Delta\rho$ is given as (Alvarez, Gaume, Polchinski and Wise, 1983; Barbieri and Maiani, 1983; Lim, Inami and Sakai, 1984)

$$\Delta\rho = \frac{3\alpha}{16\pi \sin^2\theta_W} \frac{1}{M_W^2} (m_{\tilde{u}}^2 + m_{\tilde{d}}^2 - \frac{2m_{\tilde{u}}^2 m_{\tilde{d}}^2}{m_{\tilde{u}}^2 - m_{\tilde{d}}^2} \ln \frac{m_{\tilde{u}}^2}{m_{\tilde{d}}^2}). \tag{8.17}$$

Superficially, the form of this equation is the same as that in (8.12) for $(t, b)^t$ doublet

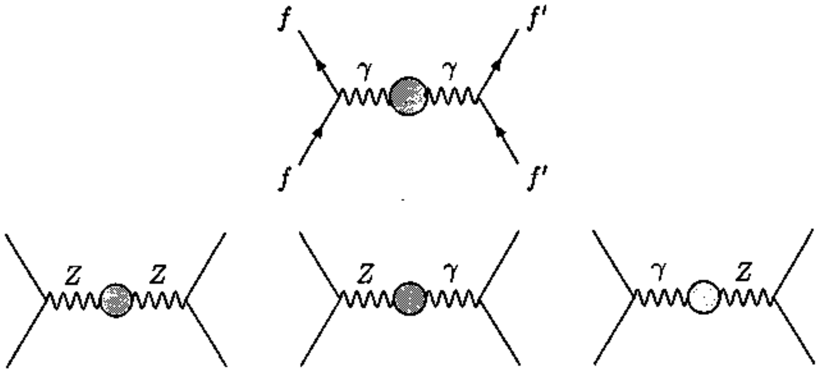


Fig. 8.5

contribution, and the decoupling of super-partners does not seem to be the case. As the matter of fact, the decoupling holds: Since $m_{u,d} \ll M_{SUSY}$, we can expand the formula for $\Delta\rho$ in terms of powers of $\frac{m_{\tilde{u},d}^2}{M_{SUSY}^2}$ to get

$$\Delta\rho \simeq \frac{\alpha}{16\pi \sin^2\theta_W} \frac{(m_u^2 - m_d^2)^2}{M_W^2\ M_{SUSY}^2}, \tag{8.18}$$

which shows the suppression by $\frac{1}{M_{SUSY}^2} \simeq \frac{1}{m_{\tilde{u},d}^2}$ and therefore the decoupling of super-partners. The factor $(m_u^2 - m_d^2)^2$ indicates that $\Delta\rho$ is described by an irrelevant operator which is quartic in the Higgs field, as we will see later in this chapter.

8.3 Oblique Corrections and S, T, U Parameters

As has been already stated in 6.1, heavy particles of the standard model, t quark and Higgs, and heavy particles predicted by New Physics do not directly appear in the external lines of Feynman diagrams for the lower energy processes of light particles. They appear only or at least mainly through their contributions to gauge boson self-energies, so called "oblique corrections". Up to the order of $O(\alpha^2)$, or up to the 1-loop order, the oblique corrections to the process $\bar{f}f \to \bar{f}'f'$ are shown in Fig. 7.5, where the blob denote the radiative corrections to gauge boson self-energies due to the heavy particles.

Such oblique corrections are known to be conveniently incorporated by "star prescription" (Kennedy and Lynn, 1989), namely by replacing the bare quantities at tree level amplitude $e, s^2(\equiv \sin^2\theta_W)$, etc. by corresponding "star" quantities $e_\star(q^2), s_\star^2(q^2)$, etc.. which incorporate quantum corrections as well and depend on q^2, with q^μ being the momentum of intermediate gauge bosons. More explicitly, the effective 4-Fermi Lagrangian for the process (in momentum space) is written as ($c_\star^2 \equiv 1 - s_\star^2$)

$$\begin{aligned}\mathcal{L}_{eff} &= e_\star^2 QQ' \, (\bar{f}\gamma_\mu f) \, \frac{1}{q^2} \, (\bar{f}'\gamma^\mu f') \\ &+ \frac{e_\star^2}{c_\star^2 s_\star^2} \, (\bar{f}\gamma_\mu [I_3 L - s_\star^2 Q] f) \, \frac{Z_\star}{q^2 - M_\star^2} \, (\bar{f}'\gamma_\mu [I_3' L - s_\star^2 Q'] f'), \quad (8.19)\end{aligned}$$

where Q, Q' and I_3, I_3' are electric charges (in the unit of e) and the third components of weak isospin of fermions f and f', respectively.

Let us demonstrate this and find the expression of the star quantities in terms of the vacuum polarization functions $\Pi(q^2)$. We start from writing the radiative corrections to gauge boson self-energies denoted by blobs in Fig. 8.5, in the form of induced quadratic terms for gauge bosons in the effective Lagrangian, in terms of the vacuum polarization functions $\Pi(q^2)$:

$$\frac{1}{2} \, \Pi_{\gamma\gamma} g_{\mu\nu} A^\mu A^\nu + \frac{1}{2} \, \Pi_{ZZ} g_{\mu\nu} Z^\mu Z^\nu + \Pi_{Z\gamma} g_{\mu\nu} Z^\mu A^\nu + \Pi_{WW} g_{\mu\nu} W^{+\mu} W^{-\nu}, \quad (8.20)$$

where we have included the correction to charged gauge boson self-energy Π_{WW}, for later use. We have ignored the part proportional to $q_\mu q_\nu$ in the vacuum polarization tensors, since in the scattering amplitude this part provides relatively negligible terms which are at most proportional to external light fermion masses after the usage of equations of motion, such as $q_\mu \bar{f}\gamma^\mu\gamma_5 f = 2m_f \, \bar{f}\gamma_5 f$. At the order of $\mathcal{O}(\alpha^2)$ or at the 1-loop level, the effective 4-Fermi Lagrangian relevant for the scattering is given as

$$\begin{aligned}\mathcal{L}_{eff} &= \left(\, eQ\bar{f}\gamma_\mu f \, , \; \tfrac{e}{cs}\bar{f}\gamma_\mu (I_3 L - s^2 Q) f \, \right) \\ &\times \begin{pmatrix} q^2 - \Pi_{\gamma\gamma} & -\Pi_{Z\gamma} \\ -\Pi_{Z\gamma} & q^2 - M_0^2 - \Pi_{ZZ} \end{pmatrix}^{-1} \begin{pmatrix} eQ'\bar{f}'\gamma^\mu f' \\ \tfrac{e}{cs}\bar{f}'\gamma^\mu (I_3' L - s^2 Q') f' \end{pmatrix} \\ &\simeq \left(\, eQ\bar{f}\gamma_\mu f \, , \; \tfrac{e}{cs}\bar{f}\gamma_\mu (I_3 L - s^2 Q) f \, \right) \\ &\times \begin{pmatrix} \frac{1}{q^2(1-\Pi'_{\gamma\gamma})} & \frac{\Pi'_{Z\gamma}}{q^2 - M_0^2} \\ \frac{\Pi'_{Z\gamma}}{q^2 - M_0^2} & \frac{1}{q^2 - M_0^2 - \Pi_{ZZ}} \end{pmatrix} \begin{pmatrix} eQ'\bar{f}'\gamma^\mu f' \\ \tfrac{e}{cs}\bar{f}'\gamma^\mu (I_3' L - s^2 Q') f' \end{pmatrix} \\ &\simeq \frac{e^2}{1 - \Pi'_{\gamma\gamma}} (Q\bar{f}\gamma_\mu f) \, \frac{1}{q^2} (Q'\bar{f}'\gamma_\mu f') \\ &+ \frac{e^2}{c^2 s^2} (\bar{f}\gamma_\mu [I_3 L - (s^2 - cs\Pi_{Z\gamma}) Q] f) \\ &\times \frac{1}{q^2 - M_0^2 - \Pi_{ZZ}} \, (\bar{f}'\gamma_\mu [I_3' L - (s^2 - cs\Pi_{Z\gamma}) Q'] f'), \quad (8.21)\end{aligned}$$

where $\Pi_{\gamma\gamma} \equiv q^2 \, \Pi'_{\gamma\gamma}$, $\Pi_{ZZ} \equiv q^2 \, \Pi'_{ZZ}$, and $M_0^2 = \frac{e^2 v^2}{4c^2 s^2}$ is the bare Z boson mass-squared. In the last step, we have used the fact that the off-diagonal element of the matrix for propagators, $\frac{\Pi'_{Z\gamma}}{q^2 - M_0^2}$, may be replaced by $\frac{\Pi'_{Z\gamma}}{q^2 - M_0^2 - \Pi_{ZZ}}$ at the 1-loop level,

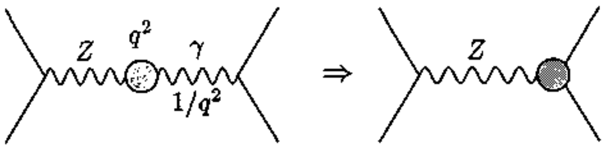


Fig. 8.6

and can be combined with the contribution of diagonal element accompanied by Π_{ZZ}. The key ingredient here is that not only $\Pi_{\gamma\gamma}$ but also $\Pi_{Z\gamma}$ is proportional to q^2 because of the CVC of QED. Namely in the diagram with $\gamma - Z$ mixing in Fig. 8.6, the factor of photon propagator $1/q^2$ is cancelled by the factor q^2 in $\Pi_{Z\gamma}$, thus making the photon propagator shrunk, yielding a amplitude exactly of the same form as that of (the vector coupling part of) pure Z-exchange diagram (see Fig. 8.6).

The factor of Z-propagator $\frac{1}{q^2 - M_0^2 - \Pi_{ZZ}}$ should have a pole at $q^2 = M_Z$, with M_Z being the well-determined physical Z boson mass: $M_Z^2 - M_0^2 - \Pi_{ZZ}(M_Z^2) = 0$, which may be used to eliminate the bare mass M_0^2 from our final result. Namely, the Taylor expansion of $\Pi_{ZZ}(q^2)$ around M_Z^2 gives an expression in terms of the physical mass: $q^2 - M_0^2 - \Pi_{ZZ}(q^2) = q^2 - M_Z^2 - (q^2 - M_Z^2)\frac{d\Pi_{ZZ}}{dq^2}|_{q^2=M_Z^2} - \Pi_{\rm res} \simeq (1 - \frac{d\Pi_{ZZ}}{dq^2}|_{q^2=M_Z^2})(q^2 - M_Z^2 - \Pi_{\rm res})$, where $\Pi_{\rm res}$ denotes the part $O((q^2 - M_Z^2)^2)$: $\Pi_{\rm res} = \Pi_{ZZ}(q^2) - \Pi_{ZZ}(M_Z^2) - (q^2 - M_Z^2)\frac{d\Pi_{ZZ}}{dq^2}|_{q^2=M_Z^2}$. The factor $(1 - \frac{d\Pi_{ZZ}}{dq^2}|_{q^2=M_Z^2})$ is the residue at the pole of the propagator.

Now the expression (8.21) takes the same form as (8.19), and we can easily read off the star-quantities:

$$
\begin{aligned}
\frac{1}{e_\star^2} &= \frac{1}{e^2}\,(1 - \Pi'_{\gamma\gamma}) \\
&\simeq \frac{1}{4\pi\alpha}\,\{1 - [\Pi'_{\gamma\gamma}(q^2) - \Pi'_{\gamma\gamma}(0)]\}, && (8.22) \\
s_\star^2 &= s^2 - cs\,\Pi'_{Z\gamma}, && (8.23) \\
Z_\star &= \frac{(\frac{e^2}{c^2 s^2})}{(\frac{e_\star^2}{c_\star^2 s_\star^2})}\,(1 + \frac{d\Pi_{ZZ}}{dq^2}|_{q^2=M_Z^2}) \\
&\simeq 1 + \frac{d\Pi_{ZZ}}{dq^2}|_{q^2=M_Z^2} - \Pi'_{\gamma\gamma} - \frac{c^2 - s^2}{cs}\,\Pi'_{Z\gamma}, && (8.24) \\
M_\star^2 &= M_Z^2 + \Pi_{\rm res} \\
&= M_Z^2 + \Pi_{ZZ}(q^2) - \Pi_{ZZ}(M_Z^2) - (q^2 - M_Z^2)\frac{d\Pi_{ZZ}}{dq^2}|_{q^2=M_Z^2}, && (8.25)
\end{aligned}
$$

where $\frac{1}{4\pi\alpha} = \frac{1}{e^2}\,[1 - \Pi'_{\gamma\gamma}(0)]$ has been used. Let us note that generally when the vacuum polarization functions $\Pi(q^2)$ are Taylor expanded in the powers of q^2, only $\Pi(0)$ and the first derivative $\frac{d\Pi}{dq^2}|_{q^2=0}$ are possibly UV-divergent, since they correspond to the radiative corrections to gauge boson mass-squared and kinetic terms. This statement is true for the Taylor expansion around arbitrary point, such as $q^2 = M_Z^2$. Thus $\Pi'_{\gamma\gamma}(q^2) - \Pi'_{\gamma\gamma}(0)$ and $\Pi_{\text{res}} = \Pi_{ZZ}(q^2) - \Pi_{ZZ}(M_Z^2) - (q^2 - M_Z^2)\frac{d\Pi_{ZZ}}{dq^2}|_{q^2=M_Z^2}$, e.g., are finite, since they contain only the terms with 2nd or higher derivatives, and we have got formulae in which the finiteness of physical quantities, like e_* and M_* are manifest . The finiteness of Z_* is less trivial (See Problem 8.4). The remaining formula for s_* contains bare quantity s^2 and is not manifestly finite. So we attempt to rewrite s^2 by a quantity made from well-measured observables and e_* at the Z pole, $\sin 2\theta_W|_Z \equiv (\frac{e_*^2(M_Z^2)}{\sqrt{2}G_F M_Z^2})^{1/2}$. From the relations

$$4\pi\alpha = e^2(1 + \Pi'_{\gamma\gamma}(0)) \tag{8.26}$$

$$M_Z^2 = \frac{e^2 v^2}{4c^2 s^2} + \Pi_{ZZ}(M_Z^2) \tag{8.27}$$

$$\frac{G_F}{\sqrt{2}} = \frac{1}{2v^2}(1 - \frac{\Pi_{WW}(0)}{c^2 M_Z^2}), \tag{8.28}$$

We get a formula for $\theta_W|_Z$

$$\begin{aligned}
\sin^2\theta_W|_Z &= s^2 + \delta\ (s^2) = s^2 + 2cs\delta\theta_W = s^2 + \frac{2c^2s^2}{c^2 - s^2}\ \delta(\ln(\sin 2\theta_W)) \\
&= s^2 + \frac{c^2 s^2}{c^2 - s^2}\ (\frac{\delta\alpha}{\alpha} - \frac{\delta G_F}{G_F} - \frac{\delta M_Z^2}{M_Z^2}) \\
&= s^2 + \frac{c^2 s^2}{c^2 - s^2}\ (\Pi'_{\gamma\gamma} + \frac{\Pi_{WW}(0)}{c^2 M_Z^2} - \frac{\Pi_{ZZ}(M_Z^2)}{M_Z^2}).
\end{aligned} \tag{8.29}$$

Thus substituting s^2 obtained from the above relation, we finally get

$$s_*^2 = \sin^2\theta_W|_Z - \frac{c^2 s^2}{c^2 - s^2}\ (\Pi'_{\gamma\gamma} + \frac{\Pi_{WW}(0)}{c^2 M_Z^2} - \frac{\Pi_{ZZ}(M_Z^2)}{M_Z^2}) - cs\ \Pi'_{Z\gamma}(q^2). \tag{8.30}$$

The part described by Π's turns out to be finite from a similar reasoning to the case of Z_* and from the property that $\Pi_{WW}(0) - c^2\Pi_{ZZ}(0)$ is finite, as it should be. All of these finite radiative corrections are the genuine predictions of the theory we consider and can be precisely tested by various experiments. For instance the corrected s_*^2 may be tested by studying parity violating asymmetries. The asymmetry of the scattering cross-sections of polarized electrons at the Z boson pole is an example, which is very sensitive to the precise value of the Weinberg angle, since the vector coupling of the Z boson to the charged lepton like electron is close to zero, $-\frac{1}{4} + \sin^2\theta_W \simeq 0$:

$$A_{LR} \equiv \frac{\sigma(e_L^- e^+ \to Z) - \sigma(e_R^- e^+ \to Z)}{\sigma(e_L^- e^+ \to Z) + \sigma(e_R^- e^+ \to Z)} = \frac{8(\frac{1}{4} - \sin^2\theta_W)}{1 + (1 - 4\sin^2\theta_W)^2} \simeq 8(\frac{1}{4} - \sin^2\theta_W). \tag{8.31}$$

The vacuum polarization functions $\Pi_{\gamma\gamma}$, $\Pi_{Z\gamma}$, Π_{ZZ} can be written, factoring the gauge coupling constants out and indicating the indices of $SU(2)_L$ and $U(1)_{em}$

adjoint representations by 1,2,3, and Q, as

$$\begin{aligned}
\Pi_{\gamma\gamma} &= e^2\,\Pi_{QQ}, &(8.32)\\
\Pi_{Z\gamma} &= \frac{e^2}{cs}\,(\Pi_{3Q} - s^2\Pi_{QQ}), &(8.33)\\
\Pi_{ZZ} &= \frac{e^2}{c^2s^2}\,(\Pi_{33} - 2s^2\Pi_{3Q} + s^4\Pi_{QQ}), &(8.34)\\
\Pi_{WW} &= \frac{e^2}{s^2}\,\Pi_{11}, &(8.35)
\end{aligned}$$

where $\Pi_{22} = \Pi_{11}$ due to the unbroken $U(1)_{em}$ symmetry, and it may also be written as $\frac{\Pi_{11}+\Pi_{22}}{2}$ as well. We can understand the form of $\Pi_{Z\gamma}$, Π_{ZZ} easily if we recall that neutral current coupled with Z takes a form $\frac{e}{cs}\,\bar{f}\gamma_\mu(I_3L - s^2Q)f$.

As an example we calculate the contributions to these polarization functions from a pair of quarks $(t,b)^t$, which may be regarded as the quarks of another generation including 4-th generation as well, if it ever exists. All vacuum polarization functions are reduced to just 2 independent functions which correspond to the vacuum polarization amplitudes induced by two electro-weak currents of quarks with the same and opposite chiralities, respectively:

$$\begin{aligned}
&\Pi_{LL}(m_1^2, m_2^2, q^2) = \Pi_{RR}(m_1^2, m_2^2, q^2)\\
&= -\frac{12}{(4\pi)^2}\int_0^1 dt\,\ln\left(\frac{\Lambda^2}{M^2 - t(1-t)q^2}\right)\cdot[t(1-t)q^2 - \frac{1}{2}M^2], \qquad (8.36)\\
&\Pi_{LR}(m_1^2, m_2^2, q^2) = \Pi_{RL}(m_1^2, m_2^2, q^2)\\
&= -\frac{12}{(4\pi)^2}\int_0^1 dt\,\ln\left(\frac{\Lambda^2}{M^2 - t(1-t)q^2}\right)\cdot\frac{1}{2}m_1m_2, \qquad (8.37)
\end{aligned}$$

where $M^2 = tm_1^2 + (1-t)m_2^2$. Writing polarization function induced by vector currents as

$$\Pi_{VV} = \Pi_{L+R,L+R} = \Pi_{LL} + \Pi_{RR} + \Pi_{LR} + \Pi_{RL} = 2(\Pi_{LL} + \Pi_{LR}), \qquad (8.38)$$

the polarization functions of our interest are now written in terms of these two functions as

$$\begin{aligned}
\Pi_{QQ}(q^2) &= Q_t^2\Pi_{VV}(m_t^2, m_t^2, q^2) + Q_b^2\Pi_{VV}(m_b^2, m_b^2, q^2), &(8.39)\\
\Pi_{3Q}(q^2) &= \frac{1}{2}Q_t\Pi_{LV}(m_t^2, m_t^2, q^2) - \frac{1}{2}Q_b\Pi_{LV}(m_b^2, m_b^2, q^2),\\
&= \frac{1}{4}[Q_t\Pi_{VV}(m_t^2, m_t^2, q^2) - Q_b\Pi_{VV}(m_b^2, m_b^2, q^2)], &(8.40)\\
\Pi_{33}(q^2) &= \frac{1}{4}[\Pi_{LL}(m_t^2, m_t^2, q^2) + \Pi_{LL}(m_b^2, m_b^2, q^2)], &(8.41)\\
\Pi_{11}(q^2) &= \frac{1}{2}\Pi_{LL}(m_t^2, m_b^2, q^2). &(8.42)
\end{aligned}$$

It is worth noting that the "chiral amplitude" has non-decoupling effect for $m_t \gg m_b$,

$$\Pi_{LL}(m_t^2, m_t^2, q^2) \simeq \frac{6}{(4\pi)^2}\, m_t^2 \ln\left(\frac{\Lambda^2}{m_t^2}\right), \tag{8.43}$$

$$\Pi_{LL}(m_t^2, m_b^2, q^2) \simeq \frac{3}{(4\pi)^2}\, m_t^2 \left(\ln\left(\frac{\Lambda^2}{m_t^2}\right) + \frac{1}{2}\right), \tag{8.44}$$

which contribute to $\Delta\rho$ as

$$\Pi_{11}(0) - \Pi_{33}(0) \simeq \frac{1}{2}\Pi_{LL}(m_t^2, m_b^2, 0) - \frac{1}{4}\Pi_{LL}(m_t^2, m_t^2, 0) \simeq \frac{3}{64\pi^2}\, m_t^2. \tag{8.45}$$

On the other hand, when Π_{LL} and Π_{LR} are summed up to get Π_{VV}, relevant for the vector-like theories such as QED or QCD, the quadratic term m_t^2 are cancelled out and Π_{VV} is proportional to q^2:

$$\Pi_{VV}(m_t^2, m_t^2, q^2) = q^2\left\{-\frac{24}{(4\pi)^2}\int_0^1 t(1-t)\ln\left(\frac{\Lambda^2}{m_t^2 - t(1-t)q^2}\right) dt\right\}. \tag{8.46}$$

This proportionality to q^2 is the result of CVC valid in the vector-like theories, and is an indication that in such theories the decoupling theorem holds.

The star quantities are functions of q^2 and each of them has an infinite number of observables as the functions of heavy particle masses appearing as the coefficients of the Taylor expansion in terms of q^2. We, however, can argue that possible non-decoupling radiative corrections, oblique corrections, of heavy particles are concentrated only in three parameters, called S, T, and U-parameters (Peskin and Takeuchi, 1990), which therefore make the analysis of New Physics simple and transparent.

To see this, suppose we have a vacuum polarization function $\Pi_{ij}(q^2)$ where the indices (i,j) take either $SU(2)$ adjoint indices $1, 2, 3$ or Q corresponding to unbroken $U(1)_{em}$. At least as far as we retain in the 1-loop level, vertices of the diagrams of vacuum polarizations are all proportional to dimensionless gauge couplings (at higher loop levels we may have mass dependent couplings due to Higgs exchanges), and we may rely on a naive dimensional analysis. Namely when we Taylor expand it as the power series in q^2 or equivalently in $\frac{q^2}{M^2}$ (M: a generic mass of intermediate heavy particles),

$$\Pi_{ij}(q^2) = \Pi_{ij}(0) + \frac{d\Pi_{ij}}{dq^2}\Big|_{q^2=0}\, q^2 + \cdots, \tag{8.47}$$

we readily know that the order of magnitudes of $\Pi_{ij}(0)$ and $\frac{d\Pi_{ij}}{dq^2}|_{q^2=0}$ are $\mathcal{O}(M^2)$ and $\mathcal{O}(q^2)$, respectively, and the residual terms are at most of $\mathcal{O}(\frac{(q^2)^2}{M^2})$, which give only decoupling effects. Thus as far as our focus is on the possible non-decoupling effects we may treat only the first two terms of the expansion. Thus, at the first sight, there seem to exist 8 ($= 2 \times 4$) parameters, corresponding to the 4 choices of the gauge indices (i,j): $(1,1) = (2,2), (3,3), (3,Q), (Q,Q)$, or equivalently the choices of gauge bosons in the external lines of the vacuum polarization diagrams $(W^+, W^-), (Z, Z), (\gamma, \gamma), (Z, \gamma)$. Actually, not all of these parameters exist. In fact, $\Pi_{QQ}(0) = \Pi_{3Q}(0) = 0$ as the result of CVC of $U(1)_{em}$: we never have mass renormalization to the photon as we wish the photon to travel at the speed of light. We thus have $8 - 2 = 6$ parameters remain. We also should aware of the fact that some of the remaining 6 parameters should be understood as to be used for the renormaliation of three bare quantities, g, g', v describing the gauge sector. This is because the first two terms of the Taylor expansion correspond to the radiative corrections to the operators with mass dimensions $d = 2$ and $d = 4$ (in "broken phase" of the theory). Thus among 6 parameters, three parameters are used for renormalization, i.e. in the process to fix the bare parameters in terms of α, G_F, M_Z etc.. In other words, these are inputs rather than the prediction of the theory. In this way, the outputs of the theory, i.e. the genuine predictions of the theory, are in the remaining $6 - 3 = 3$ parameters, which are nothing but the S, T, U-parameters we are interested in.

As is clear from the above argument, such three parameters S,T,U should be automatically finite quantities, as they correspond to operators, which do not exist in the original Lagrangian: they need not to be renormalized. Actually, such finite combinations of poralization functions have already appeared in the discussion of star-quantities, i.e. in the attempt to show the finiteness of the star-quantities. Namely the parameters are defined as follows:

$$\alpha S \equiv 4e^2\, [\Pi'_{33}(0) - \Pi'_{3Q}(0)], \tag{8.48}$$

$$\alpha T \equiv \frac{e^2}{c^2 s^2 M_Z^2}\, [\Pi_{11}(0) - \Pi_{33}(0)], \tag{8.49}$$

$$\alpha U \equiv 4e^2\, [\Pi'_{11}(0) - \Pi'_{33}(0)], \tag{8.50}$$

where we have used a notation $\Pi'_{ij}(0)$ to denote $\frac{d\Pi_{ij}}{dq^2}|_{q^2=0}$, which are nearly equal to $\Pi_{ij}(q^2)/q^2$ for the case of $\Pi_{ij}(0) = 0$, which were originally denoted as Π'_{ij}, once we ignore $\mathcal{O}(\frac{(q^2)^2}{M^2})$. We can confirm that S, T, U really do not suffer from the UV-divergences. The finiteness of S-parameter is guaranteed by the fact that there is no mixing term of $SU(2)_L$ and $U(1)_Y$ field strengths, $F^i_{\mu\nu}\ B^{\mu\nu}$ in the original Lagrangian. T and U should not get UV-divergence since whose existence contradicts with the "custodial symmetry" of the terms $\sum_{i=1}^3\ F^i_{\mu\nu}\ F^{i\mu\nu}$ and $\frac{g^2 v^2}{4}\ \sum_{i=1}^3 A^i_\mu A^{i\mu}$ in the original Lagrangian: no couterterm can be prepared for the quantities which violate the symmetry of the original Lagrangian.

We can easily check that, in the approximation that $\mathcal{O}(\frac{(q^2)^2}{M^2})$ is negligible, the star quantities are now describable in terms of S, T, U-parameters:

$$\frac{1}{e_*^2} - \frac{1}{4\pi\alpha} \simeq 0, \tag{8.51}$$

$$s_*^2 - \sin^2\theta_W|_Z \simeq \frac{\alpha}{c^2 - s^2}\, [\frac{1}{4}S - c^2 s^2\ T], \tag{8.52}$$

$$
\begin{aligned}
Z_\star - 1 &\simeq \frac{\alpha}{4c^2s^2}\, S, && (8.53)\\
M_\star^2 - M_Z^2 &\simeq 0, && (8.54)
\end{aligned}
$$

where $e_\star$ and $M_\star$ do not obtain quantum effects in this approximation, i.e. for $q^2 \ll M^2$, since these quantum effects stem from 2nd or higher derivatives of Π_{ij}. In addition, we can list up a related star-quantity $\rho_\star(0)$ which is defined as the ratio of the 4-Fermi coupling of low energy ($q^2 = 0$) neutral current process to that of charged current process:

$$\mathcal{L}_{eff}^{(NC)} \equiv -\frac{8G_F}{\sqrt{2}}\, \rho_\star(0)(\bar{f}\gamma_\mu[I_3L - s_\star^2 Q]f)(\bar{f}'\gamma_\mu[I_3'L - s_\star^2 Q']f'). \qquad (8.55)$$

After some arithmetic we find

$$\rho_\star(0) - 1 \simeq \alpha\, T. \qquad (8.56)$$

Thus we know that the T-parameter (times α) is equivalent to $\Delta\rho$ we had been discussing.

As the example we consider the contribution of a pair of quarks (t, b) (or equivalently that of (t', b'), the hypothetical quarks of 4th generation) to the S, T parameters. By use of (8.48), (8.49), (8.39)-(8.42) and (8.36)-(8.38), we easily find their contributions to be:

$$
\begin{aligned}
S &= \frac{1}{2\pi}\,[1 - \frac{2}{3}\ln\frac{m_t}{m_b}], && (8.57)\\
T &= \frac{3}{16\pi}\,\frac{1}{c^2s^2}\,\frac{1}{M_Z^2}\,[m_t^2 + m_b^2 - \frac{2m_t^2m_b^2}{m_t^2 - m_b^2}\ln\frac{m_t^2}{m_b^2}], && (8.58)
\end{aligned}
$$

where the formula for T is just $\Delta\rho/\alpha$, as we expected. We learn that in the limit of global $SU(2)$ "custodial" (or weak isospin) symmetry, i.e. $m_t = m_b$, T vanishes, just as $\Delta\rho$ does, but (the first term of) S remains: $S = \frac{1}{2\pi}$.

This result is relevant especially for the technifermion contribution, since in technicolor theory (Susskind, 1979; Weinberg, 1979), just as in QCD, the technifermion condensation, corresponding to the VEV of Higgs, preserves the custodial symmetry: $<\bar{T_U}T_U> = <\bar{T_D}T_D>$, which means that the resultant dynamical masses due to technicolor interaction are degenerated, and each technifermion doublet contributes to S-parameter additively. Thus the precision test on S parameter is expected to yield a meaningful constraint on the theory. To be precise the above calculation cannot be directly applied to the contributions of bound states such as techni-hadrons, since technifermions are treated as free fermions in the above analysis. Thus more sophisticated approach is desirable, though the above results are enough for rough estimation. A sophisticated analysis, relying on the dispersion relations of vacuum polarization functions of vector- and axial-vector types, whose imaginary parts are measurable R-ratios, provides

$$S = 0.4 \text{ (for 1 technifermion doublet)}$$

$$= \quad 2.1 \text{ (for 1 "generation" techinicolor model)}, \qquad (8.59)$$

assuming that the number of the technicolor $N_{TC} = 4$. Comparing with the upper bound on S obtained from the precision measurements at LEP etc., Peskin and Takeuchi were able to rule out the 1 "generation" technicolor model (Peskin and Takeuchi, 1990).

The constraints on S, T, U-parameters imposed by the precision experiments at LEP etc. are not only used to get information on the new physics, but also quite useful to constrain the masses of heavy particles in the standard model, i.e. m_t and m_H. Especially, since the T-parameter ($\Delta\rho$) is quite sensitive to m_t^2, it was possible to predict rather precisely the value of m_t as somewhere around $175(GeV)$, before the direct discovery of the t quark at Fermi Lab.

8.4 Global Symmetries

We have seen that all of S, T, U parameters are all "calculable" quantities, free from UV-divergence without any need of renormalization. ¿From a bit different point of view this is the consequences of global symmetries, which remain in the gauge sector of the theory even after the SSB. To see this, let us take a matrix form of the Higgs fields and introduce a global $SU(2)_R$ symmetry.

We start by writing the Yukawa coupling in the following form with the matrix form Φ of the Higgs field (ignoring its VEV for a while),

$$\begin{aligned} \mathcal{L}_{Yukawa} &= f_u \begin{pmatrix} \bar{u_L}, & \bar{d_L} \end{pmatrix} \begin{pmatrix} \varphi^0 \\ \varphi^- \end{pmatrix} u_R + f_d \begin{pmatrix} \bar{u_L}, & \bar{d_L} \end{pmatrix} \begin{pmatrix} -\varphi^+ \\ \varphi^{0*} \end{pmatrix} d_R + h.c. \\ &= \begin{pmatrix} \bar{u_L}, & \bar{d_L} \end{pmatrix} \Phi \begin{pmatrix} f_u & 0 \\ 0 & f_d \end{pmatrix} \begin{pmatrix} u_R \\ d_R \end{pmatrix} + h.c., \end{aligned} \qquad (8.60)$$

where

$$\Phi \equiv \begin{pmatrix} \varphi^0 & -\varphi^+ \\ \varphi^- & \varphi^{0*} \end{pmatrix} = \begin{pmatrix} \frac{H+iG^3}{\sqrt{2}} & -iG^+ \\ iG^- & \frac{H-iG^3}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} [HI + iG^i \sigma^i], \qquad (8.61)$$

which just corresponds to the linear-σ model discussed in QCD to describe the low energy effective theory, where H and G^i are replaced by a scalar and pseudo-scalar bound states of quarks, σ and π^i ($i = 1, 2, 3$). The technicolor theory takes this analogy seriously and regard the Higgs fields as the bound states of hypothetical fermions with quantum numbers "technicolors", instead of ordinary colors, called technifermions T. In particular $H \sim \bar{T_U} T_U + \bar{T_D} T_D$ (for 1 doublet model of technicolor), though the Λ_{TC}, the mass scale where the technicolor interaction becomes strong, is scaled up to $\mathcal{O}(1TeV)$.

If the degeneracy $m_u = m_d$ is realized, the theory is invariant under a global chiral symmetry $SU(2)_L \times SU(2)_R$:

$$\begin{aligned} \begin{pmatrix} u_L \\ d_L \end{pmatrix} &\to \begin{pmatrix} u'_L \\ d'_L \end{pmatrix} = g_L \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \begin{pmatrix} u_R \\ d_R \end{pmatrix} \to \begin{pmatrix} u'_R \\ d'_R \end{pmatrix} = g_R \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \\ \Phi &\to \Phi' = g_L \ \Phi \ g_R^*. \\ & (\quad g_L, \ g_R : \text{elements of } \mathrm{SU}(2)_L \text{ and } \mathrm{SU}(2)_R) \end{aligned} \tag{8.62}$$

Let us note the squared absolute-value of the Higgs doublet is given in this matrix form as $\mathrm{Tr}(\Phi^\dagger \ \Phi) = H^2 + G^{i2}$, which is clearly invariant under the $SU(2)_L \times SU(2)_R$: $\mathrm{Tr}(\Phi'^\dagger \ \Phi') = \mathrm{Tr}(\Phi^\dagger \ \Phi) \ : \ H'^2 + G'^{i2} = H^2 + G^{i2}$. This means the real fields H, G^i behave as the fundamental representation of $SO(4)$, which is equivalent to $SU(2) \times SU(2)$, as is well-known in the group theory. Just as in QCD, after Higgs develops its VEV v ($H \to v + H$), corresponding to non-vanishing $<\bar{u}u>=<\bar{d}d>$ in QCD, the chiral symmetry is spontaneously broken, but leaving its vector-like subgroup $SU(2)_V$ generated by $g_L = g_R \equiv g$ unbroken:

$$g^\dagger \begin{pmatrix} \frac{v}{\sqrt{2}} & 0 \\ 0 & \frac{v}{\sqrt{2}} \end{pmatrix} g = \begin{pmatrix} \frac{v}{\sqrt{2}} & 0 \\ 0 & \frac{v}{\sqrt{2}} \end{pmatrix}. \tag{8.63}$$

This remaining vector-like symmetry is called "custodial symmetry" (Sikivie et al., 1980), or we may just call it as (global) weak isospin symmetry. Let us note that such custodial symmetry is valid only when the degeneracy of the quark masses $m_u = m_d$ is realized, since

$$g^\dagger \begin{pmatrix} f_u & 0 \\ 0 & f_d \end{pmatrix} g \neq \begin{pmatrix} f_u & 0 \\ 0 & f_d \end{pmatrix}, \tag{8.64}$$

unless $f_u = f_d$. This is intuitively trivial as $m_u \neq m_d$ should break the weak isospin symmetry.

The $SU(2)_L$ gauge bosons A^i_μ ($i = 1, 2, 3$) and $\mathrm{U}(1)_Y$ gauge boson B^μ transform under the chiral symmetry $SU(2)_L \times SU(2)_R$ and the custodial symmetry as

$$(A^1_\mu, A^2_\mu, A^3_\mu) : \quad (3, 1) \ \text{ and } \ 3 \tag{8.65}$$

$$B^\mu : \quad (1, 3) + (1, 1) \ \text{ and } \ 3 + 1, \tag{8.66}$$

where the representation of the $U(1)_Y$ gauge boson breaks into the two pieces $(1, 3)$ and $(1, 1)$, since the weak-hypercharge may be written as $\frac{Y}{2} = I_{3R} + \frac{B-L}{2}$, where I_{3R} is the eigenvalue of the 3rd component of "right-handed weak-isospin" $SU(2)_R$ and B, L denote baryon and lepton numbers respectively. (Or we may rewrite the Nakano–Nishijima–Gell-Mann relation as $Q = I_{3L} + I_{3R} + \frac{B-L}{2}$ in a left-right symmetric form, which really holds in the left-right symmetric gauge theory $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$, though in the gauge group of the standard model, we are taking now, the $SU(2)_R$ is not gauged.)

Paying attention to the fact that $\Pi'_{33} - \Pi'_{3Q} = \frac{1}{2}\Pi'_{3Y}$ denotes the mixing of $SU(2)_L$ and $U(1)_Y$ gauge bosons in the kinetic term, and that $\Pi_{11} - \Pi_{33} = \frac{1}{2}[\Pi_{11} +$

$\Pi_{22}] - \Pi_{33}$ we realize the representation of the oblique parameters under the chiral and custodial symmetries:

$$S: \quad (3,3) + (3,1) \quad 1 + 3 + 5 \tag{8.67}$$

$$T, U: \quad (5,1) \quad 5. \tag{8.68}$$

where in the representation of S, $1 + 5$ comes from (3.3) while 3 comes from (3.1). Let us recall the simple algebra in $SU(2)$ group theory, $3 \times 3 = 1 + 3 + 5$, and note that only 1 and 5 are relevant for us, since vacuum polarization functions should be symmetric under the exchange of gauge indices. The linear combination $\frac{1}{2}[\Pi_{11} + \Pi_{22}] - \Pi_{33}$ can be expressed by a traceless 3×3 matrix and should belong to 5 representation of $SU(2)$, while traceful combination $\Pi_{11} + \Pi_{22} + \Pi_{33}$ behave as the singlet. It is now easy to understand that T, U-parameters should vanish in the limit of custodial symmetry: T, U behave as a non-singlet of $SU(2)_V$, namely as 5, and should vanish in the limit of exact $SU(2)_V$ symmetry, since in the all orders of quantum corrections, quantities contradicting with the symmetry never appear. In such sense, the symmetry handling the T, U or $\Delta\rho$ is the custodial symmetry. Strictly speaking, it may be more suitable to say that actually what handle these parameters is the global $SU(2)_L$ rather than the custodial symmetry (Inami-Lim-Yamada, 1992) simply because these parameters are concerned only with $SU(2)_L$. (In fact to get 5 representation we need quartic term of quark masses, or equivalently an operator quartic in the Higgs field, as we will see in the following subsection, while the quark mass matrix $M_q = \frac{v}{\sqrt{2}}\mathrm{diag}(f_u, f_d)$ has a piece of 3 under $SU(2)_V$ and quadratic term of quark mass is enough to get 5, if we invoke to the custodial symmetry. This becomes clear in the operator analysis extended in the next section.) Then, what symmetry handles the S-parameter ? This time obviously custodial symmetry is not the one, since the part $(3,3)$ of S contains a singlet 1 under the $SU(2)_V$, the constant term $\frac{1}{2\pi}$ in (8.57), which survives even in the limit of degenerate doublet ($m_t = m_b$). We may say that the relevant symmetry for the parameter S is just the chiral symmetry $SU(2)_L \times SU(2)_R$, as the S-parameter behaves as $(3,3)$, therefore not as an invariant under the symmetry (Inami, Lim and Yamada, 1992). In this way we can readily understand why each of the technifermion doublet gives additive contribution to S: each of the condensation $<\bar{T_U}T_U> = <\bar{T_D}T_D>$ break the chiral symmetry spontaneously, thus contributing to S.

8.5 Operator Analysis

We have seen T, for instance, is calculable finite quantity, although the gauge boson mass-squared seems to be the coefficient of relevant operators with $d = 2$, W^+W^- etc. As was discussed in section 8.1, in the renormalizable theory a calculable quantity should be a coefficient of some irrelevant operator. It is known that the T-parameter is described by the coefficient of the following irrelevant operator (Problem 8.6)

$$O_T = (\phi^\dagger\ D_\mu\ \phi)(\phi^\dagger\ D^\mu\ \phi) - \frac{1}{3}\ (\phi^\dagger\ D_\mu D^\mu\ \phi)(\phi^\dagger \phi). \tag{8.69}$$

Similarly, S parameter is proportional to the coefficient of another $d = 6$ operator,

$$O_S = [\phi^\dagger\ (F^i_{\mu\nu}\sigma^i)\ \phi]\ B^{\mu\nu}. \tag{8.70}$$

It is easy to see that when the replacement $\phi \to (0, \frac{v}{\sqrt{2}})^t$ is made this operator causes the mixing between the field strengths of $SU(2)_L$ and $U(1)_Y$, i.e. $F^3_{\mu\nu}$ and $B^{\mu\nu}$.

It is now obvious why the S, T, U-parameters are all calculable finite quantities: they are all described by the coefficients of higher dimensional $d > 4$ irrelevant operators, such as those listed above, which do not exist in the original Lagrangian. Thus the coefficients should be all finite as long as the theory is renormalizable. Now the difference of decoupling and non-decoupling effects of heavy particles is also easy to understand. The examples we discussed above tell us that when heavy particles' contributions to physical observables are decoupling ones, there is a characteristic feature that heavy particles get their masses through some new mass scales, such as M_{SUSY}, which are inevitably gauge invariant. In that case, the coefficients of the irrelevant operators induced by a heavy particle will be suppressed by the inverse powers of M, with M denoting a generic new large mass scale. For instance the effective Lagrangian for the S-parameter behave as

$$\sim \frac{1}{M^2}\ [\phi^\dagger\ (W^i_{\mu\nu}\sigma^i)\phi]\ B^{\mu\nu}, \tag{8.71}$$

as is clear from dimensional analysis. This means the heavy particle contribution is suppressed by the factor $1/M^2$. On the other hand, we have learned that when heavy particles' contributions are non-decoupling ones, heavy particles get their masses through "strong" couplings with Higgs, the origin of the masses being the VEV of the Higgs (SSB), without demanding any new large mass scales. In this case M_W is a unique mass scale and the coefficients of irrelevant operators are not suppressed, and even enhanced by the "strong" couplings in some suitably chosen cases. This is why such contributions are non-decoupling.

One defect of the operator analysis extended above is that we may put arbitrary powers of a gauge invariant factor $\phi^\dagger\phi$ to a given irrelevant operator, all of them giving the same operator in the broken phase. When we are considering some decoupling effects, it will not be a real problem, since the insertion of $\phi^\dagger\phi$ will make the coefficients of the operators more suppressed by higher inverse powers of M, such as $1/M^4, 1/M^6$, which are less important. In the case of non-decoupling effects, however, such further suppression is not the case, as $(\phi^\dagger\phi)/v^2 \ \to\ O(1)$ after the replacement of the Higgs by its VEV. This problem is evaded once we move to the non-linear realization of the Higgs field, as discussed in section 8.2:

$$U \equiv e^{i\frac{G^i\sigma^i}{v}}, \tag{8.72}$$

where the physical Higgs field H has been ignored, since it does not appear in the processes we are interested in (at 1-loop level). In this non-linear representation we do not have the problem of getting infinite number of operators for one observable, since $U^\dagger U = I$ is a c-number. Analysis based on the non-linear realization shows that potential non-decoupling effects in gauge boson 2 and 3 point functions are given by 7 independent operators. The reason why we can treat both of the 2 and 3 point functions on an equal footing is that in non-Abelian gauge theory they are mutually related. For instance, O_S not only has gauge boson 2-point function, but also 3-point function as well, since $F^i_{\mu\nu}$ has both linear and quadratic terms of gauge bosons.

Problems

8.1 Calculate the diagrams in Fig.8.3 to get the formula for $\Delta\rho$ in (8.12). Also show that (8.12) vanishes for $m_t = m_b$.

8.2 Show that (8.17) is approximated by (8.18) when $m^2_{u,d} \ll M^2_{SUSY}$ $(m^2_{\tilde{u}} = m^2_u + M^2_{SUSY}, m^2_{\tilde{d}} = m^2_d + M^2_{SUSY})$.

8.3 Verify the relation in (8.21) at the 1-loop level.

8.4 Prove that Z_* defined in (8.25) is finite.

8.5 Verify the formula (8.57) by use of (8.40), (8.41) and (8.48).

8.6 Show that the T-parameter is given as the coefficient of the operator O_T in (8.69).

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Chapter 9

FLAVOR PHYSICS AND CP VIOLATION

9.1 The Interest in Flavor Physics and CP Violation

The gauge sector of the standard model (SM) is theoretically uniquely determined by gauge principle. The Higgs sector, on the other hand, has many arbitrary parameters, whose values cannot be theoretically predicted. Especially so far we have no definite idea about the origin of quark and lepton masses and generation or flavor mixings, i.e. the origin of the Yukawa couplings of Higgs field. Issues related with quark and lepton flavors, "flavor physics", is therefore very important clue, not only to the the confirmation of the standard model, but also to the search for some theories beyond the standard model, "New Physics", where the origin of flavors is expected to be understood more deeply.

Among such flavor physics, Flavor Changing Neutral Current (FCNC) processes are of special interest. They do not exist at tree or classical level in the standard model, and are induced only at loop or quantum level. Therefore the observation of such FCNC processes will provide us valuable information on the contributions of all particles as the intermediate states. They are, therefore, very suitable, not only to get information about the flavor mixings of relatively heavy t quark with lower generations, but also to search for possible heavy particle effects of New Physics. Namely the studies into the FCNC processes is important for the progress of particle physics. The rates of such FCNC processes are known to be suppressed by both of the higher order of the perturbation theory and the small mass difference or small flavor mixings. Thus, the FCNC processes are often called rare processes.

In fact, historically the rare processes have played very important role in the foundation of the particle physics. We may list up some of the most important events, some of them have already been discussed in 4.6.2: the introduction of c quark, in order to naturally suppress the rare processes of neutral kaon such as $K^0 \leftrightarrow \bar{K}^0$ mixing, by GIM (Glashow, Iliopoulos and Maiani, 1970), the prediction of the mass of such predicted c quark by Gaillard and Lee (Gaillard and Lee, 1974), the introduction of the third generation to implement the observed CP violation in the neutral kaon system by Kobayashi and Maskawa (Kobayashi and Maskawa, 1973), which embodies the standard model nowadays, the lower bound on m_t imposed by the data on $B^0 \leftrightarrow \bar{B}^0$ mixings, obtained before the direct discovery of t quark, etc.

The CP-violating processes are also known to happen rather rarely in neutral kaon system. In such sense, the physics of CP violation also provides valuable information of the heavy t quark and/or heavy unknown particles of New Physics. We should further note that the Kobayashi-Maskawa (KM) model of 3 generations was originally devised as the theory to accommodate CP violation. In spite of the remarkable success of the KM model, such as $N_\nu = 3$ (N_ν: the number of light neutrinos) at LEP, yet there has been no final conclusive argument on the origin of CP violation. So, the confirmation of the prediction on CP violating observables of the model, such as CP asymmetries in B decays, is an urgent necessity to really establish the standard model. We may even find some evidence of some New Physics in the course of the investigation.

The FCNC processes and CP violation is, in principle, mutually independent. In fact, in the model of spontaneous (or soft) CP violation (Lee, 1974; Weinberg, 1976), the CP violation is attributed to the complex phase of the vacuum expectation value (VEV) of the Higgs created by the suitably chosen scalar potential, and has no direct relation with the flavor mixing. On the other hand, in the mechanism of KM, both of quark mass matrix causing the flavor mixing and CP violation come from the same Yukawa couplings. Therefore they are mutually very closely related. In fact in KM model we will see below that the CP violation needs mass differences and flavor mixing, (of course) in addition to the complex phase in the KM matrix. CP violation, therefore, is observed in FCNC processes, such as $K^0 \leftrightarrow \bar{K}^0$ and $B^0 \leftrightarrow \bar{B}^0$ mixings.

The purpose of this chapter is to discuss the rare processes due to FCNC and the CP violation. We assume the theory we work on is the standard model, since to provide the results in SM is quite useful even if we further investigate New Physics. Furthermore, some of the formulae we derive are readily applicable for the class of New Physics, where the Yukawa coupling of the Higgs has the same structure as that of the standard model, e.g. the model with four or more generations, minimal supersymmetric stadard model. In this chapter we will discuss FCNC rare processes only in the quark sector. Once neutrinos become massive, the lepton flavor violation, i.e. FCNC processes in lepton sector, also becomes physically meaningful. We, however, will not deal with the lepton flavor violation here, since the neutrino oscillation as the typical example of lepton flavor violation has been extensively discussed in chapter 6.

9.2 Flavor Symmetry and FCNC Rare Processes

As was discussed in 4.6.2, GIM escaped FCNC processes at tree level, by introducing c quark. What GIM has proposed is that by introducing c quark, the weak isospin of s quark became the same as that of d quark and even after the uni-

tary transformation to the mass eigenstates the neutral current is kept flavor- or generation-diagonal. We may say that the essence of their idea is to make the gauge interaction of each sector of fermions with definite electric charge and chirality universal, i.e. to make it invariant under global $SU(n_g)$ transformation, with n_g being the number of the generations ($n_g = 3$ in KM model). If we call the gauge interaction to connect the elements of doublet "vertical symmetry", this flavor (or generation) symmetry may be called as "horizontal symmetry". By imposing such global flavor symmetry, the FCNC, such as $\bar{s}\gamma_\mu d$ coupled with Z boson or photon are strictly forbidden at the tree level. We call such flavor conservation in the neutral currents as "natural flavor conservation". The concept of natural is used to denote some property which is naturally guaranteed by some symmetry of the theory irrespectively of the tuning of the parameters of the theory. From such point of view, for instance, small Higgs mass in the standard model is not natural (the hierarchy problem), but it can be naturally small in the SUSY standard model, etc. Glashow and Weinberg have summarized the conditions for the natural flavor conservation in neutral currents coupled with gauge or Higgs bosons (Glashow–Weinberg, 1977):
1. Fermions with the same electric charge and chirality should belong to the same representation of the gauge group.
2. Fermions with definite electric charge should couple with only one Higgs doublet.

The first condition is the condition for the neutral current gauge interactions do not have FCNC. If the condition is met, a neutral current $J_\mu^{(0)}$ generally has a form in the base of weak-eigenstates

$$\begin{aligned} & (\overline{u_{1L}}, \overline{u_{2L}}, \ldots, \overline{u_{1R}}, \overline{u_{2R}}, \ldots, \overline{d_{1L}}, \overline{d_{2L}}, \ldots, \overline{d_{1R}}, \overline{d_{2R}}, \ldots) \\ \times \quad & \mathrm{diag}(a, a, \ldots, b, b, \ldots, c, c, \ldots, d, d, \ldots) \\ \times \quad & (u_{1L}, u_{2L}, \ldots, u_{1R}, u_{2R}, \ldots, d_{1L}, d_{2L}, \ldots, d_{1R}, d_{2R}, \ldots)^t, \end{aligned} \tag{9.1}$$

and this form is invariant under the unitary transformations to the mass eigenstates, since the transformation is among the fields with the same electric charge and chirality, such as $u_{1L}, u_{2L}, \ldots$, and therefore the unitary matrix for the transformation takes the form of block-diagonal:

$$U_L^\dagger \mathrm{diag}(a, a, \ldots) U_L = \mathrm{diag}(a, a, \ldots), \tag{9.2}$$

etc. The second condition is necessary for the Yukawa coupling to naturally conserve flavors. If, for instance, up-type quarks are allowed to couple with two Higgs doublets ϕ_1, ϕ_2, their neutral Higgs fields generally have the following Yukawa couplings,

$$\overline{u_{iL}}[f_{ij}^{(1)}\varphi_1^0 + f_{ij}^{(2)}\varphi_2^0]\ u_{jR} + h.c.. \tag{9.3}$$

One linear combination of the Higgs fields, $\frac{1}{\sqrt{v_1^2+v_2^2}}[v_1\ \varphi_1^0 + v_2\ \varphi_2^0]$, $(<\varphi_{1,2}^0>= \frac{v_{1,2}}{\sqrt{2}})$, has a flavor-diagonal Yukawa coupling with the mass eigenstates $u, c, \ldots$ in accordance with the diagonalization of the mass matrix for up-type quarks, while

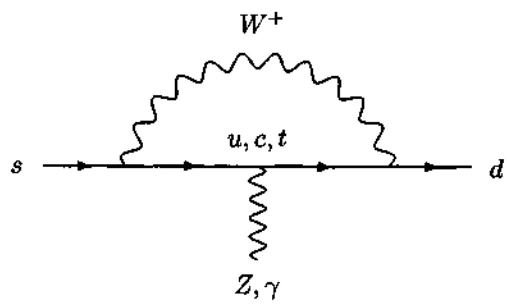

Fig. 9.1

the Yukawa coupling of the orthogonal linear combination $\frac{1}{\sqrt{v_1^2+v_2^2}}[v_2\ \varphi_1^0 - v_1\ \varphi_2^0]$ causes the FCNC at the tree level. Let us note that in SUSY standard model, because of the chirality (holomorphic property) of the super-potential, even though we have two Higgs doublets, there exists a selection rule to forbid the simultaneous Yukawa couplings of the two doublet Higgs.

Since the standard model satisfies the Glashow–Weinberg's condition, the FCNC is forbidden at the tree level. The flavor symmetry $SU(n_g)$, however, is broken by the quark mass differences in each of up- and down-type quark sectors, i.e. by $m_u \neq m_c \neq m_t$ etc. We may still have symmetries of each generation number, i.e. a sub-symmetry $U(1)^{n_g}$, if there is no generation mixings, but we know that there exist such mixings and the flavor symmetry is thus completely broken. We thus expect that at the quantum or loop level of Feynman diagram the FCNC processes are induced by charged current interactions, which have generation mixings via KM matrix U. In Fig. 9.1, we have a diagram with W^+-exchange, which induces effective $\bar{d}sZ$ and $\bar{d}s\gamma$ vertices.

The amplitude of Fig. 9.1 is expected to vanish for an exact $SU(3)$ symmetry, $m_u = m_c = m_t$. In fact, we easily find this is the case, since in this case the orthogonality of the KM-matrix $\sum_i U_{id}^* U_{is} = 0$ works. Thus, to get FCNC we need the violation of flavor symmetry due to both of non-degenerate quark masses at up- (and down-) type quark sector, and flavor mixings. More intuitively we may say that when the degeneracy $m_u = m_c = m_t$ holds there is no mean to distinguish the (u, c, t) quarks. Thus we may treat (u', c', t') obtained by arbitrary unitary transformation of (u, c, t) as mass eigenstates with an equal right. Thus it is always possible to choose the unitary transformation of the left-handed up-type quarks, so that it makes the KM-matrix unit matrix, thus making the generation mixing meaningless. Suppose we have only two generations with a mixing angle θ_c. Then the above amplitude is proportional to $U_{cd}^* U_{cs}\ \frac{m_c^2 - m_u^2}{M_W^2} = \cos\theta_c\ \sin\theta_c\ \frac{m_c^2 - m_u^2}{M_W^2} \simeq 6 \times 10^{-5}$. We learn from this simple exercise that the amplitudes of FCNC processes are suppressed by small mass-squared differences (the GIM mechanism). If we include the third generation the relevant mass differences $m_t^2 - m_{u,c}^2$ are not small compared with M_W^2. But, on the other hand, the mixings of t with 1st or 2nd generations are small compared with $\sin\theta_c$, thus making the amplitude rather small. Thus, the rates of FCNC processes are tiny and these processes are often called "rare processes". It is worth noticing that the FCNC processes in quark sector and the neutrino oscillation in lepton sector are both induced by the mass differences of different generations and generation mixing. This is not surprising, as the neutrino oscillation is one of FCNC processes.

9.3 Rare Processes in Kaon System

As we have seen in the introductory argument in section 9.1, the rare processes in neutral kaon system of $(K^0, \bar{K^0})$ have played very important roles in the establishment of the standard model. $(K^0, \bar{K^0})$ are the bound states of s and d quarks and their anti-quarks, $K^0 \sim \bar{s}\gamma_5 d$, $\bar{K^0} \sim \bar{d}\gamma_5 s$. There is a freedom of relative phase of these two states, but here we adopt a convention $CP\ |K^0\rangle = -|\bar{K^0}\rangle$. We may define a quantum number (a $U(1)$ charge) "strangeness" S carried only by s and $\bar{s}$ quarks, $S = -1$ for s, $S = 1$ for $\bar{s}$, $S = 0$ otherwise. The neutral kaons, K^0, $\bar{K^0}$, thus have $S = -1,\ 1$, respectively. In the FCNC processes of kaons, the strangeness changes, and we may classify the rare processes by the extent of the change of the strangeness:
(a) $|\Delta S| = 2$: $K^0 \leftrightarrow \bar{K^0}$,
(b) $|\Delta S| = 1$: $K_L \to \mu\bar{\mu}$, $K^+ \to \pi^+\nu\bar{\nu}$,
where in the decay $K^+ \to \pi^+\nu\bar{\nu}$, the contributions of neutrinos of all generations should be summed up, as long as they are light. We discuss these two types of FCNC processes separately below.

(a) $|\Delta S| = 2$ process

The CPT theorem tells us that the masses and the life time (total decay width) of particle and its anti-particle should be exactly the same. Thus if only strong interaction QCD is switched on, K^0 and $\bar{K^0}$, being anti-particles of one another, are two degenerate states without any interaction (as the strong interaction preserves the strangeness S). The situation is somehow similar to the case of double-well potential with very high wall separating two degenerate vacuum states $|1\rangle$, $|2\rangle$ in the quantum mechanics (see Fig. 9.2).

In the case of double-well potential, as the height of the barrier wall gets reduced the tunneling effects connecting two states $|1\rangle$, $|2\rangle$ starts to operative. Then the Hamiltonian in the base of these states takes a form

$$H = \begin{pmatrix} E_0 & E_t \\ E_t & E_0 \end{pmatrix}. \tag{9.4}$$

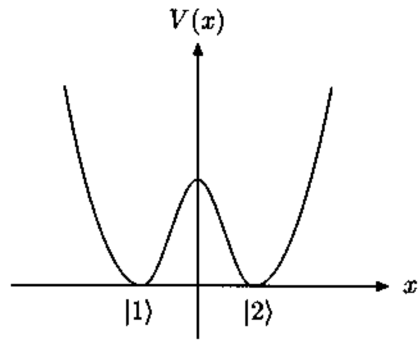

Fig. 9.2

Even if the tunneling effect E_t is small, the energy eigenstates get large modification,

$$\frac{1}{\sqrt{2}}\,[|1\rangle - |2\rangle], \quad \frac{1}{\sqrt{2}}\,[|1\rangle + |2\rangle], \tag{9.5}$$

with energy eigenvalues $E_0 - E_t$ and $E_0 + E_t$.

Similarly, when weak interaction is switched on to the neutral kaon system, the strangeness is no longer preserved and FCNC process $K^0 \leftrightarrow \bar{K^0}$ arises. The Hamiltonian in the base of $(K^0, \bar{K^0})$ at the rest frame now reads as

$$H = \begin{pmatrix} M & M_{12} \\ M_{12}^* & M \end{pmatrix}, \tag{9.6}$$

where M is the common mass implied by the CPT theorem and M_{12} is due to the $K^0 \leftrightarrow \bar{K^0}$ mixing. For simplicity, here we have ignored absorptive part, denoted by Γ, coming from the on-shell intermediate states, which correspond to the fact that the neutral kaons are unstable states decaying into $\pi\pi$ etc. Thus the Hamiltonian is hermitian, though we will treat non-hermitian Hamiltonian including the absorptive part later in this section. We further simplify by ignoring a small CP violating effect in the element M_{12} due to the CP phase of KM matrix for a while, assuming $M_{12}^* = M_{12}$. Then just as in the case of quantum mechanics, we find two mass (energy) eigenstates

$$|K_1\rangle \equiv \frac{1}{\sqrt{2}}\,[|K^0\rangle - |\bar{K^0}\rangle], \quad |K_2\rangle \equiv \frac{1}{\sqrt{2}}\,[|K^0\rangle + |\bar{K^0}\rangle], \tag{9.7}$$

with the mass eigenvalues

$$m_1 = M - M_{12}, \quad m_2 = M + M_{12}. \tag{9.8}$$

As we have ignored the possible CP violating effect, these states have definite CP eigenvalues:

$$CP\;|K_1\rangle = |K_1\rangle, \quad CP\;|K_2\rangle = -|K_2\rangle. \tag{9.9}$$

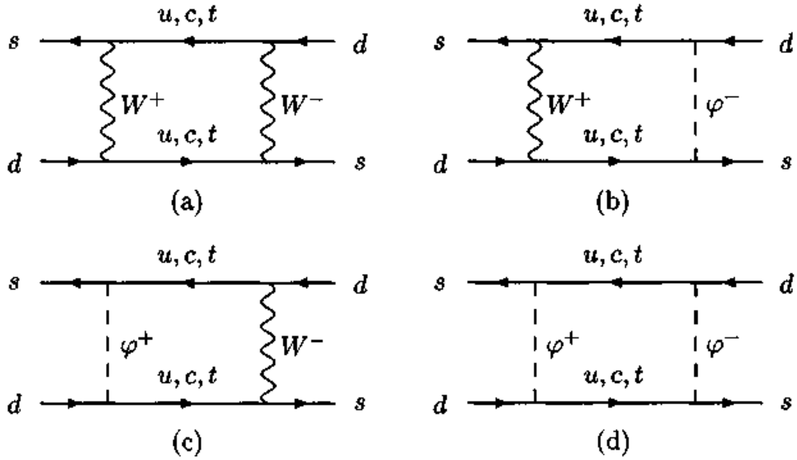

Fig. 9.3

Actually when we include the CP violating effect and the absorptive parts, real eigenstates of the Hamiltonian are K_S and K_L states with shorter and longer life times, which are nearly equal to K_1 and K_2, respectively. Their mass difference

$$\Delta m_K \equiv m_{K_L} - m_{K_S} \simeq 2M_{12}, \tag{9.10}$$

is an observable, and is known to be extremely small: $\Delta m_K = (3.552 \pm 0.016) \times 10^{-12}$ MeV.

The elementary process to cause the mixing at the quark level is the 1-loop diagram, called "box diagram" (See Fig. 9.3).

In Fig. 9.3, $\varphi^{\pm}$ are the would-be Nambu–Goldstone boson, which has a propagator in R_ξ gauge (discussed in the gauge theories with spontaneous gauge symmetry breaking), $\frac{i}{k^2 - \frac{M_W^2}{\xi} + i\epsilon}$. As we see below the effective Lagrangian due to the top quark exchange in this box diagram has a term proportional to m_t^2, which comes from the diagram with φ^+ and φ^- exchanges (at least for $\xi \neq 0$), which has 4 Yukawa couplings yielding a factor $(gm_t/M_W)^4$. The factor is then multiplied by $1/{m_t}^2$ (for $m_t \geq M_W$) coming from dimensional analysis, thus producing a net effect $(g^4 m_t^2)/M_W^4$, a non-decoupling effect of heavy t quark. Note that this box diagram is only possible quark diagram at 1-loop level; the diagrams with 1-loop induced FCNC Z- or γ (photon)-vertices, which are relevant for $|\Delta S| = 1$ processes, such as $K_L \to \mu\bar{\mu}$, changes the strangeness only by 1 unit and do no contribute to the $|\Delta S| = 2$ process $K^0 \leftrightarrow \bar{K^0}$. In the pioneering calculation by Gaillard–Lee, an assumption that the masses of intermediate up-type quarks are much smaller than M_W was made, which is a good approximation in the 2 generation (4 quark) model, $m_c \ll M_W$, which was the standard picture of that time. The calculation, however, needs to be revised, since we now know that there are 3 generations of quarks and m_t is greater than the weak-scale M_W. The calculation which is valid for the arbitrary masses of intermediate up-type quarks was performed (Inami and Lim, 1981) and the result can be represented in a form of effective 4-Fermi $|\Delta S| = 2$ operator (see also (8.14)):

$$\mathcal{L}_{eff}^{|\Delta S|=2} = \frac{G_F}{\sqrt{2}} \frac{\alpha}{4\pi \sin^2 \theta_W} \sum_{i,j=c,t} (V_{is}^* V_{id})(V_{js}^* V_{jd}) \ E(x_i, x_j) \ (\bar{s}\gamma_\mu L d)(\bar{s}\gamma^\mu L d), \tag{9.11}$$

where the contributions of two internal quarks $u_i, \ u_j \ (i, j = u, c, t)$ are contained in the coefficient function (with $x_i \equiv \frac{m_i^2}{M_W^2}$)

$$\begin{aligned} E(x_i, x_j) &= -x_i x_j \{ \frac{1}{x_i - x_j} [(\frac{1}{4} - \frac{3}{2} \frac{1}{x_i - 1} - \frac{3}{4} \frac{1}{(x_i - 1)^2}) \ \ln x_i - (x_i \to x_j)] \\ &- \frac{3}{4} \frac{1}{(x_i - 1)(x_j - 1)} \}, \end{aligned} \tag{9.12}$$

and $L = \frac{1-\gamma_5}{2}$, V_{is} etc. are K-M matrix elements. We have replaced the original coefficient function $\hat{E}(x_i, x_j)$ by $E(x_i, x_j) = \hat{E}(x_i, x_j) - \hat{E}(x_u, x_j) - \hat{E}(x_i, x_u) + \hat{E}(x_u, x_u)$, setting $x_u = \frac{m_u^2}{M_W^2} = 0$. The additional three terms exactly vanish by the orthogonality of K-M matrix $\sum_i V_{is}^* V_{id} = 0$, $\sum_j V_{js}^* V_{jd} = 0$. This is why the summation of i, j is only over c and t. When both intermediate quarks coincide we get

$$E(x_i) \equiv E(x_i, x_i) = -\frac{3}{2} (\frac{x_i}{x_i - 1})^3 \ \ln x_i - x_i [\frac{1}{4} - \frac{9}{4} \frac{1}{x_i - 1} - \frac{3}{2} \frac{1}{(x_i - 1)^2}]. \tag{9.13}$$

Though the above result is valid for an arbitrary intermediate quark masses, the external quark masses m_d, m_s and their 4-momenta have been ignored there, which is a good approximation. To get M_{12}, we need to evaluate the matrix element of the 4-Fermi effective operator with respect to kaon states,

$$\langle K^0 | (\bar{s}\gamma_\mu L d)(\bar{s}\gamma^\mu L d) | \bar{K^0} \rangle = \frac{2}{3} \ f_K^2 m_K^2 B, \tag{9.14}$$

where f_K, m_K are decay constant and the mass of K-meson ($m_K \simeq M$ in (9.6)), and the B is the "bag-parameter", which parameterizes the ambiguity due to the non-perturbative QCD effects to form the bound states K^0 and $\bar{K^0}$, and the reference value $B = 1$ corresponds to the vacuum saturation, $\langle K^0 | (\bar{s}\gamma_\mu L d) | 0 \rangle \langle 0 | (\bar{s}\gamma^\mu L d) | \bar{K^0} \rangle$. Thus writing the off-diagonal element of the neutral kaon mass-squared matrix as

$$\begin{aligned} \delta m^2 &\equiv -\langle \bar{K^0} | \mathcal{L}_{eff}^{|\Delta S|=2} | K^0 \rangle \\ &= -\frac{G_F}{\sqrt{2}} \frac{\alpha}{6\pi \sin^2 \theta_W} f_K^2 m_K^2 B \ [(V_{is}^* V_{id})(V_{js}^* V_{jd}) \ E(x_i, x_j)], \end{aligned} \tag{9.15}$$

the mass-squared matrix reads as

$$\begin{pmatrix} M^2 & \delta m^2 \\ \delta m^2 & M^2 \end{pmatrix}, \tag{9.16}$$

and taking the square-root we obtain the Hamiltonian at the rest frame, with an approximation $\delta m^2 \ll M^2$,

$$H \simeq \begin{pmatrix} M & \frac{\delta m^2}{2M} \\ \frac{\delta m^2}{2M} & M^2 \end{pmatrix}, \tag{9.17}$$

which means $M_{12} \simeq \frac{\delta m^2}{2M}$. We thus obtain

$$\begin{aligned} \Delta m_K &\simeq 2\mathrm{Re}M_{12} \simeq \frac{\mathrm{Re}\delta m^2}{M} \\ &\simeq -\frac{G_F}{\sqrt{2}}\frac{\alpha}{6\pi\sin^2\theta_W} f_K^2 m_K B\ \mathrm{Re}[(V_{is}^* V_{id})(V_{js}^* V_{jd})]\ E(x_i, x_j) \\ &\simeq -3.8\times 10^{-2} m_K B\ \mathrm{Re}[(V_{is}^* V_{id})(V_{js}^* V_{jd})]\ E(x_i, x_j). \end{aligned} \tag{9.18}$$

In the case of restricted 2 generation model, noting $E(x_c) \simeq -x_c$ for $x_c \ll 1$, we get

$$\Delta m_K \simeq \frac{\alpha G_F f_K^2 m_K}{6\sqrt{2}\pi\sin^2\theta_W} B \frac{(\sin\theta_c\cos\theta_c)^2\ m_c^2}{M_W^2}, \tag{9.19}$$

which reduces to the result obtained by Gaillard and Lee. Comparing this with the observed Δm_K they succeeded to predict the mass of c quark to be $m_c \simeq 1.3$(GeV), before the discovery of the c quark. Unfortunately, in the full 3 generation model, the generation mixings of t quark with lower generations have not been determined precisely, thus making the prediction of m_t not straightforward. In addition, though the above result is suitable for perturbative contributions of heavier quarks c, t, there may exist non-perturbative effects of intermediate u quark, which make our prediction not conclusive. The CP violation observable ϵ discussed later, however, being proportional to $\mathrm{Im}(V_{is}^* V_{id} V_{js}^* V_{jd})$, does not suffer from such problem, since the intermediate u quark does not contribute, and the above formula is directly applicable. Thus by use of ϵ we can impose a useful constraint on the generation mixings and CP phase, namely on the unitarity triangle discussed later in this chapter.

(b) $|\Delta S| = 1$ processes

The typical $|\Delta S| = 1$ processes are $K_L \to \mu\bar{\mu}$ and $K^+ \to \pi^+\nu\bar{\nu}$. ($K_L \to \pi^0 e^+ e^-$ is the signature of CP violation and is another interesting process to study. But the derivation of its decay rate is obtained in a similar manner with the argument below, and will not be given.)

Now the diagrams with 1-loop induced FCNC Z- or γ (photon)-vertices also contribute to the processes, and the quark Feynman diagrams, relevant for the $K_L \to \mu\bar{\mu}$ and $K^+ \to \pi^+\nu\bar{\nu}$, are shown in Fig. 9.4, where ν_l denotes all possible light neutrinos ($l = e, \mu, \tau$ for the standard model). Note that the γ-exchange diagram obviously does not contribute to the decay $K^+ \to \pi^+\nu\bar{\nu}$. The 1-loop induced FCNC Z- or γ (photon)-vertices, denoted by blobs in Fig. 9.4, are the sum of the sub-diagrams shown in Fig. 9.5.

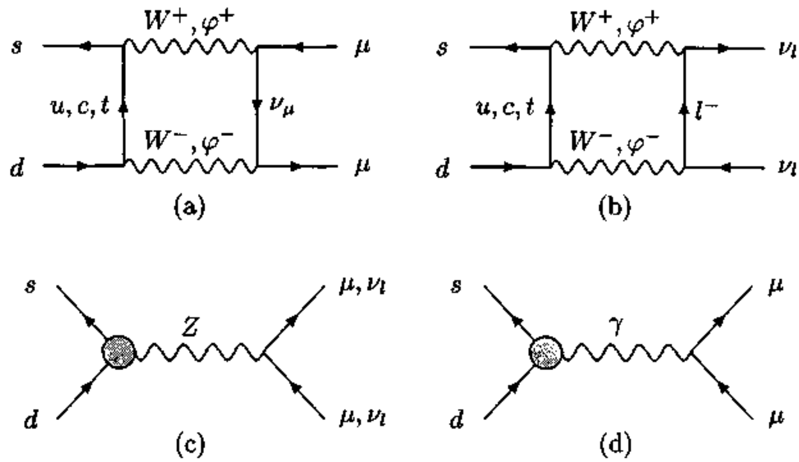

Fig. 9.4

One may wonder why we should include the 1-particle reducible diagrams with FCNC self-energies such as Fig. 9.5(a), and wonder if such FCNC self-energy should be eliminated by adding a suitable counterterm for the external quarks d, s to have physical meaning as the eigenstates of propagation. Such procedure is also possible, but it turns out that the contributions of all counterterms just cancel with each other in the FCNC processes (Botella and Lim, 1986), intuitively because the counterterms for the FCNC quark self energy and for the FCNC vertices of Z or γ gauge bosons are not independent.

We first display the results for the 1-loop induced effective FCNC Z- and γ-vertices (Inami and Lim, 1981):

$$\mathcal{L}_{\bar{s}dZ} = \frac{\alpha}{4\pi \sin^2\theta_W} \frac{g}{\cos\theta_W} \sum_{i=c,t} (V_{is}^* V_{id}) \Gamma^Z(x_i)(\bar{s_L}\gamma_\mu d_L)\, Z^\mu, \tag{9.20}$$

$$\begin{aligned}\mathcal{L}_{\bar{s}d\gamma} &= \frac{\alpha}{4\pi \sin^2\theta_W} \frac{e}{2M_W^2} \sum_{i=c,t} (V_{is}^* V_{id}) \\ &\quad \bar{s}[F_1(x_i)(q^2\gamma_\mu - q^\mu \not{q})L + F_2(x_i)\sigma_{\mu\nu} i q^\nu (m_s L + m_d R)]d\, A^\mu, \end{aligned} \tag{9.21}$$

where the effective $\bar{s}d\gamma$ vertex is given in the momentum space ($q^\mu \equiv p_s^\mu - p_d^\mu$). The coefficient functions are given in terms of the interemediate up-type quark masses $x_i = m_i^2/M_W^2$ ($i = u, c, t$) as

$$\Gamma^Z(x_i) = [\frac{1}{4} - \frac{3}{8}\frac{1}{x_i - 1}]x_i + \frac{3}{8}\frac{2x_i^2 - x_i}{(x_i - 1)^2}\ln x_i + \gamma(x_i, \xi), \tag{9.22}$$

$$\begin{aligned} F_1(x_i) &= Q\{[\frac{1}{12}\frac{1}{x_i - 1} + \frac{13}{12}\frac{1}{(x_i-1)^2} - \frac{1}{2}\frac{1}{(x_i - 1)^3}]\, x_i \\ &+ [\frac{2}{3}\frac{1}{x_i - 1} - \frac{2}{3}\frac{1}{(x_i-1)^2} - \frac{5}{6}\frac{1}{(x_i-1)^3} + \frac{1}{2}\frac{1}{(x_i-1)^4}]\, x_i \ln x_i - \frac{2}{3}\ln(\frac{x_i}{x_u})\} \end{aligned}$$

$$
\begin{aligned}
&- && [\frac{7}{3}\frac{1}{x_i-1}+\frac{13}{12}\frac{1}{(x_i-1)^2}-\frac{1}{2}\frac{1}{(x_i-1)^3}]\,x_i \\
&- && [\frac{1}{6}\frac{1}{x_i-1}-\frac{35}{12}\frac{1}{(x_i-1)^2}-\frac{5}{6}\frac{1}{(x_i-1)^3}+\frac{1}{2}\frac{1}{(x_i-1)^4}]\,x_i\ln x_i \\
&- && 2\gamma(x_i,\xi), \qquad (9.23) \\
F_2(x_i) &= && -Q\{[-\frac{1}{4}\frac{1}{x_i-1}+\frac{3}{4}\frac{1}{(x_i-1)^2}+\frac{3}{2}\frac{1}{(x_i-1)^3}]\,x_i-\frac{3}{2}\frac{x_i^2}{(x_i-1)^4}\ln x_i\} \\
&+ && [\frac{1}{2}\frac{1}{x_i-1}+\frac{9}{4}\frac{1}{(x_i-1)^2}+\frac{3}{2}\frac{1}{(x_i-1)^3}]\,x_i-\frac{3}{2}\frac{x_i^3}{(x_i-1)^4}\ln x_i, \qquad (9.24)
\end{aligned}
$$

where the term $\ln(\frac{x_i}{x_u})$ in $F_1(x_i)$ has an infra-red singularity at the limit $x_u, m_u \to 0$, and x_u dependence has been kept only there. This term just corresponds to the contribution of "penguin-diagram" with gluon-exchange (see Fig. 9.6) discussed below (we may call the logarithmic contribution as the contribution of "electro-weak penguin" diagram). In $F_{1,2}(x_i)$, Q denotes the electric charge of the internal up-type quarks, i.e. $Q = \frac{2}{3}$. We have left the factor Q free, so that we can clearly distinguish the contributions of the different types of diagrams where the photon is attached to the internal or external quarks, respectively. This makes, for instance, the calculation of $\bar{s}dg$ (g : gluon) vertex, namely the calculation of the gluon-penguin-diagram, really straightforward (Problem 9.2). The appearance of the ξ-dependent term

$$
\begin{aligned}
\gamma(x,\xi) &= -\frac{x}{8(\xi x-1)}+\frac{1}{\xi x-1}(\frac{3}{4}\frac{1}{x-1}+\frac{1}{8}\frac{1}{\xi x-1})x\ln x \\
&+ \frac{1}{8}[-\frac{(5\xi+1)x}{(\xi x-1)(\xi-1)}+\frac{2x-\xi x^2}{(\xi x-1)^2}]\ln\xi \qquad (9.25)
\end{aligned}
$$

implies that the effective vertices are not gauge invariant, though the Pauli-term accompanied by F_2 is gauge invariant, since only this term contributes to the decay $s \to d\gamma$ with the on-shell condition $q^2 = 0$. The remaining gauge dependence is cancelled when the contributions from box diagrams are also summed up, as we will see below.

Let us note that the $\bar{s}d\gamma$ vertex has two pieces, the terms with the currents $[\bar{s}(q^2\gamma_\mu - q^\mu \not{q})d]$ and $(\bar{s}\sigma_{\mu\nu}d)q^\nu$, called "charge radius" and "Pauli" terms, respectively. These currents automatically vanish (without the use of equations of motion), when multiplied by q^μ, thus satisfying the current conservation of electromagnetic current (CVC). Note that in the FCNC vertices the current conservation is not satisfied by use of the equations of motion, since s and d have different masses, but satisfied by these specific forms of the currents including the external momentum q^μ. Thus the $\bar{s}d\gamma$ vertex vanish in the limit of $q_\mu = 0$, and we need to calculate the diagram keeping the $\mathcal{O}(q^2)$ terms, in contrast to the case of calculations of other FCNC effective Lagrangians. This proportionality to q^2 makes the non-decoupling effect of heavy t quark rather mild in the $\bar{s}d\gamma$ vertex, i.e. ln (m_t/M_W), while in

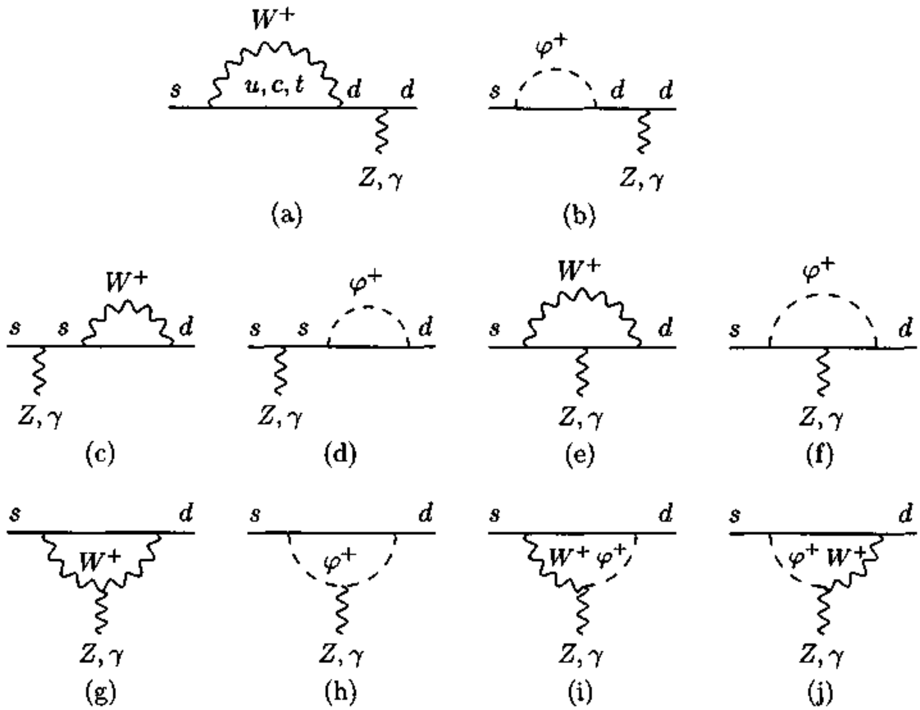

Fig. 9.5

$\bar{s}dZ$ vertex we have an non-decoupling effect proportional to m_t^2, just as in the box diagram (see (9.22)–(9.24)). The reason is rather easy to see; in the $\bar{s}dZ$ vertex the φ^+-exchange gives the Yukawa coupling factor m_t/M_W twice, thus giving the non-decoupling effect proportional to m_t^2, while in the $\bar{s}d\gamma$ vertex, the additional q^2 factor enforces the remaining piece to have mass-dimension $d = -2$, behaving as $1/m_t^2$, thus cancelling the m_t^2 from the Yukawa couplings. In this way, when the FCNC γ vertex or similarly FCNC gluon vertex is relevant, the contributions from the lighter $u.c$ quarks become significant. Especially the penguin-diagram shown in Fig. 9.6 with FCNC gluon vertex is expected to play an important role in the explanation of "$\Delta I = \frac{1}{2}$ rule" in $K \to \pi\pi$ decays.

Write the effective 4-Fermi Lagrangian relevant for the decays of our interest as

$$
\begin{aligned}
\mathcal{L}_{eff}^{|\Delta S|=1} &= \frac{4G_F}{\sqrt{2}} \frac{\alpha}{4\pi \sin^2\theta_W} \sum_{i=c,t} (V_{is}^* V_{id})[C(x_i)(\overline{s_L}\gamma_\mu d_L)(\overline{\mu_L}\gamma^\mu \mu_L) \\
&- \sum_{j=1}^{3} D(x_i, y_j)(\overline{s_L}\gamma_\mu d_L)(\overline{\nu_{jL}}\gamma^\mu \nu_{jL})], \qquad (9.26)
\end{aligned}
$$

where $y_j = \frac{m_{l_j}^2}{M_W^2}$ with m_{l_j} denoting the charged lepton mass of the j-th generation.

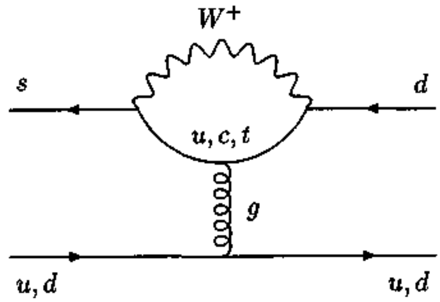

Fig. 9.6

The coefficient functions are the sum of the contributions from the box diagrams and the Z-exchange diagrams in Fig. 9.4:

$$C(x_i) = C_\square(x_i) + C_Z(x_i), \quad D(x_i, y_j) = D_\square(x_i, y_j) + D_Z(x_i). \tag{9.27}$$

We readily know that $C_Z(x_i) = D_Z(x_i) = \Gamma^Z(x_i)$. The contributions from the box diagrams are known to be

$$\begin{aligned} C_\square(x_i) &= \frac{3}{8}\frac{x_i}{(x_i-1)^2}\ln x_i - \frac{3}{8}\frac{x_i}{x_i-1} - \gamma(x,\xi) & (9.28) \\ D_\square(x_i, y_j) &= -\frac{1}{8}\frac{1}{y_j - x_i}\left(\frac{y_j-4}{y_j-1}\right)^2 x_i \ln y_j \\ &+ \frac{1}{8}\left[\frac{x_i}{y_j-x_i}\left(\frac{x_i-4}{x_i-1}\right)^2 + \frac{x_i-7}{x_i-1}\right]x_i\ln x_i \\ &- \frac{9}{8}\frac{1}{y_j-1}\frac{x_i}{x_i-1} - \gamma(x,\xi). & (9.29) \end{aligned}$$

Combining these two types of contributions we finally get gauge invariant coefficient functions

$$\begin{aligned} C(x_i) &= \frac{3}{4}\left(\frac{x_i}{x_i-1}\right)^2 \ln x_i + \frac{1}{4}x_i - \frac{3}{4}\frac{x_i}{x_i-1}, & (9.30) \\ D(x_i, y_j) &= -\frac{1}{8}\frac{y_j x_i}{y_j-x_i}\left(\frac{y_j-4}{y_j-1}\right)^2 \ln y_j \\ &+ \frac{1}{8}\left[\frac{x_i}{y_j-x_i}\left(\frac{x_i-4}{x_i-1}\right)^2 + 1 + 3\frac{1}{(x_i-1)^2}\right]x_i\ln x_i \\ &+ \frac{1}{4}x_i - \frac{3}{8}\left(1 + 3\frac{1}{y_j-1}\right)\frac{x_i}{x_i-1}. & (9.31) \end{aligned}$$

Note that the piece with vector current $\bar{\mu}\gamma_\mu\mu$ does not contributes to the $K_L \to \mu\bar{\mu}$ decay. This is because when q_μ coming from the hadronic matrix element $\langle 0|\bar{s_L}\gamma_\mu d_L|K_L\rangle = -\frac{1}{2}\langle 0|\bar{s}\gamma_\mu\gamma_5 d|K_L\rangle$ is multiplied to the leptonic current, the CVC implies $q_\mu(\bar{\mu}\gamma^\mu\mu) = 0$. Thus the photon-exchange diagram in Fig. 9.4 does not contribute to the process $K_L \to \mu\bar{\mu}$, as the matter of fact.

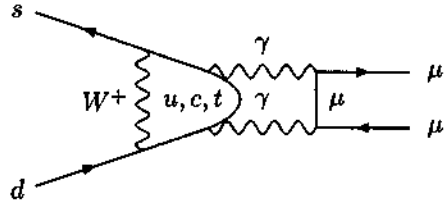

Fig. 9.7

We thus obtain formulae for the ratios of the branching ratios of two processes to those of reference decay processes, in terms of the coefficient functions given above,

$$\frac{Br(K_L \to \mu\bar{\mu})_{sd}}{Br(K^+ \to \bar{\mu}\nu_\mu)} \simeq \frac{\tau(K_L)}{\tau(K^+)} 4(\frac{\alpha}{4\pi \sin^2\theta_W})^2 \frac{[\mathrm{Re}(\sum_{i=c,t} V_{is}^* V_{id} C(x_i))]^2}{V_{us}^2}, \tag{9.32}$$

$$\frac{Br(K^+ \to \pi^+\nu\bar{\nu})}{Br(K^+ \to \pi^0 e^+ \nu_e)} = 2(\frac{\alpha}{4\pi \sin^2\theta_W})^2 \sum_j \frac{|\sum_{i=c,t} V_{is}^* V_{id} D(x_i, y_j)|^2}{(V_{us})^2}, \tag{9.33}$$

where the subscript "*sd*" denotes the short distance (or perturbative) contribution to the process, without including the two-photon process discussed below, and in the estimation of $K_L \to \mu\bar{\mu}$ the difference of phase spaces of K_L and K^+ decays have been ignored. In the case of $K_L \to \mu\bar{\mu}$, there is an additional contribution from FCNC two photon vertex, as shown in Fig. 9.7, whose presence makes the precise prediction of the branching ratio difficult, though including the two photon process the above formula gives a reasonable prediction, compatible with the observed value $Br(K_L \to \mu\bar{\mu}) = (7.25 \pm 0.16) \times 10^{-9}$.

On the other hand, the branching ratio $Br(K^+ \to \pi^+\nu\bar{\nu})$ can be reliably estimated, as the photon process does not contribute to this decay. When the values of quark masses and generation mixing angles (or KM matrix elements) are substituted and an approximation $y_j = 0$ is used, i.e.,

$$D(x_i) \equiv D(x_i, 0) = \frac{3}{4}\frac{x_i - 2}{(x_i - 1)^2} x_i \ln x_i + \frac{x_i}{4} + \frac{3}{4}\frac{x_i}{x_i - 1}, \tag{9.34}$$

the above formula gives $Br(K^+ \to \pi^+\nu\bar{\nu})$ of the order of $O(10^{-10})$, which seems to be quite consistent with the recent result of BNL experiment, which has claimed the observation of the rare decay.

We learn from (9.7) that K^0 or $\bar{K^0}$ is the mixture of K_1 and K_2, or more precisely K_S and K_L, with mass difference Δm_K due to the $K^0 \leftrightarrow \bar{K^0}$ mixing. Hence, if we can prepare an initial state of K^0 or $\bar{K^0}$, it's existence probability will oscillate in time due to the interference effects of matter waves of K_S and K_L, just as the neutrino oscillation discussed in chapter 6. A feature not shared by the neutrino oscillation is that neutral kaons decay into $\pi\pi$ etc., when they propagate. Thus the time evolution Hamiltonian have (effective) imaginary or non-hermitian part to effectively represent the decay. In order to accommodate the absorptive part and possible CP violating effects we correct (9.6) as

$$H = \begin{pmatrix} M - \frac{i}{2}\Gamma & M_{12} - \frac{i}{2}\Gamma_{12} \\ M_{12}^* - \frac{i}{2}\Gamma_{12}^* & M - \frac{i}{2}\Gamma \end{pmatrix}, \tag{9.35}$$

where M, Γ, corresponding to the mass and the decay width of neutral mesons in the absence of the mixing, are real, while M_{12} and Γ_{12} may have imaginary parts due to CP violation. Though this Hamiltonian is applicable to both of neutral kaon and neutral B-meson system ($B_d \sim \bar{b}\gamma_5 d, \bar{B}_d \sim \bar{d}\gamma_5 b; B_s \sim \bar{b}\gamma_5 s, \bar{B}_s \sim \bar{s}\gamma_5 b$), here we will discuss the neutral B meson oscillation. (Here we discuss only $(B_d, \bar{B}_d)$ system, and will simply use the abbreviation B^0 for B_d, unless otherwise mentioned.) One reason for this is that in the neutral kaon system the lifetimes of K_L and K_S are too different for the interference to be operative, while in the B meson system the difference of lifetimes due to Γ_{12} is very small.

The eigenstates of the Hamiltonian

$$|B_1\rangle \equiv \frac{1}{\sqrt{|p|^2 + |q|^2}} (p|B^0\rangle - q|\bar{B^0}\rangle), \tag{9.36}$$

$$|B_2\rangle \equiv \frac{1}{|p|^2 + |q|^2} (p|B^0\rangle + q|\bar{B^0}\rangle), \tag{9.37}$$

where

$$\frac{q}{p} = \frac{\sqrt{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}}{\sqrt{M_{12} - \frac{i}{2}\Gamma_{12}}}, \tag{9.38}$$

are known to have eigenvalues

$$\lambda_{1,2} = m_{1,2} - \frac{i}{2}\,\Gamma_{1,2}, \tag{9.39}$$

$$\Delta\lambda = \lambda_2 - \lambda_1 = \Delta m - \frac{i}{2}\,\Delta\Gamma$$

$$(\Delta m \equiv m_2 - m_1,\ \Delta\Gamma \equiv \Gamma_2 - \Gamma_1), \tag{9.40}$$

with

$$\Delta m - \frac{i}{2}\,\Delta\Gamma = 2\,\sqrt{M_{12} - \frac{i}{2}\Gamma_{12}}\sqrt{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}. \tag{9.41}$$

The time evolution of a state $\psi(t) = c(t)\,|B^0\rangle + \bar{c}(t)\,|\bar{B^0}\rangle$ is governed by a Schrödinger equation

$$i\frac{d}{dt}\begin{pmatrix} c(t) \\ \bar{c}(t) \end{pmatrix} = H \begin{pmatrix} c(t) \\ \bar{c}(t) \end{pmatrix}, \tag{9.42}$$

where the Hamiltonian H has been given in (9.35). The diagonalization of the Hamiltonian

$$\frac{\sqrt{|p|^2+|q|^2}}{2pq}\begin{pmatrix} q & -p \\ q & p \end{pmatrix} H \frac{1}{\sqrt{|p|^2+|q|^2}}\begin{pmatrix} p & p \\ -q & q \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{9.43}$$

enables us to solve the differential equation as

$$\begin{aligned} \begin{pmatrix} c(t) \\ \bar{c}(t) \end{pmatrix} &= \begin{pmatrix} p & p \\ -q & q \end{pmatrix}\begin{pmatrix} e^{-i\lambda_1 t} & 0 \\ 0 & e^{-i\lambda_2 t} \end{pmatrix}\begin{pmatrix} \frac{1}{2p} & -\frac{1}{2q} \\ \frac{1}{2p} & \frac{1}{2q} \end{pmatrix}\begin{pmatrix} c(0) \\ \bar{c}(0) \end{pmatrix} \\ &= \begin{pmatrix} g_+(t) & -\frac{p}{q}g_-(t) \\ -\frac{q}{p}g_-(t) & g_+(t) \end{pmatrix}\begin{pmatrix} c(0) \\ \bar{c}(0) \end{pmatrix}, \end{aligned} \tag{9.44}$$

where

$$\begin{aligned} g_\pm(t) &\equiv \frac{1}{2}\,[e^{-i\lambda_1 t} \pm e^{-i\lambda_2 t}] \\ &= \frac{1}{2}e^{-\frac{\Gamma_1}{2}t}e^{-im_1 t}\,[1 \pm e^{-\frac{\Delta\Gamma}{2}t}e^{-i\Delta m t}]. \end{aligned} \tag{9.45}$$

Let $|B(t)\rangle$ and $|\bar{B}(t)\rangle$ be the states with the initial conditions $|B(0)\rangle = |B^0\rangle$, $|\bar{B}(0)\rangle = |\bar{B^0}\rangle$. As $|B^0\rangle$, e.g. corresponds to $c(0) = 1, \bar{c}(0) = 0$ we easily find

$$|B(t)\rangle = g_+(t)|B^0\rangle - \frac{q}{p}g_-(t)|\bar{B^0}\rangle, \tag{9.46}$$

$$|\bar{B}(t)\rangle = g_+(t)|\bar{B^0}\rangle - \frac{p}{q}g_-(t)|B^0\rangle. \tag{9.47}$$

The "survival" and transition probabilities for a state starting as B^0 at time 0 to be found as B^0 and $\bar{B^0}$ at time t are easily found to be

$$P(B^0 \to B^0) = |\langle B^0|B(t)\rangle|^2 = |g_+(t)|^2 \simeq e^{-\Gamma t}\cos^2(\frac{\Delta m}{2}t), \tag{9.48}$$

$$P(B^0 \to \bar{B^0}) = |\langle \bar{B^0}|B(t)\rangle|^2 = |\frac{q}{p}g_-(t)|^2 \simeq |\frac{q}{p}|^2 e^{-\Gamma t}\sin^2(\frac{\Delta m}{2}t), \tag{9.49}$$

where the specific property of neutral B meson system $\Delta\Gamma \ll \Gamma_{1,2}$, $\Gamma_1 \simeq \Gamma_2 \simeq \Gamma$ has been used. Note that in neutral kaon system $\Delta\Gamma \sim \Gamma$, which causes very different life-times for the two eigenstates, K_L, K_S. Why does such difference in two systems appear? In the B meson system, the generation mixings of t quark with lower generations are not suppressed compared with the mixing with the 3rd generation, $|V_{ub}^* V_{ud}| \sim |V_{cb}^* V_{cd}| \sim |V_{tb}^* V_{td}|$, as we will see below. On the other hand the, Δm, $\Delta\Gamma$, produced by the dispersive and absorptive parts of box diagrams, M_{12}, Γ_{12}, as shown in Fig. 9.8, get contributions from intermediate t and u, c quarks, respectively.

In the case of absorptive part only u, c intermediate states are kinematically allowed to be on-shell. In the case of dispersive part all of u, c, t contribute, in principle. We, however, know that the coefficient function $E(x_i)$ of (9.13) grows up as $-\frac{1}{4}x_i = -\frac{1}{4}\frac{m_i^2}{M_W^2}$ (non-decoupling effect) for $x_i \gg 1$ and the t quark gives the

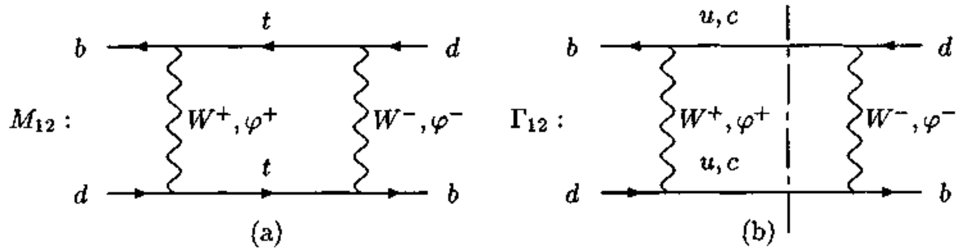

Fig. 9.8

dominant contribution. Thus the absorptive part is suppressed compared with the dispersive part by $O(m_b^2/m_t^2) \ll 1$. This is the reason why $\Delta\Gamma$ is negligible in the B system. The formulae (9.48), (9.49) remind us of those in neutrino oscillation (with maximal mixing). One remarkable difference is that in the $B^0 \leftrightarrow \bar{B^0}$ oscillation, the sum of the probabilities decrease as the time goes on, which is the consequence of the loss of the unitarity due to the non-hermitian Hamiltonian. So, when we study observables sensitive to the oscillation, such as CP asymmetries in B^0, $\bar{B^0}$ decays, $\Delta m \times \tau = \frac{\Delta m}{\Gamma} \equiv x$ needs to be sizable, which is satisfied in B_d^0, $\bar{B_d^0}$ system, where x is known to be $\simeq 0.7$. (For the B_s^0, $\bar{B_s^0}$ system, x is much larger, relatively enhanced by $|V_{ts}/V_{td}|^2$.)

We may define the ratio of the time-integrated transition and survival probabilities

$$r \equiv \frac{\int_0^\infty P(B^0 \to \bar{B^0})dt}{\int_0^\infty P(B^0 \to B^0)dt} = \frac{x^2}{2+x^2}, \tag{9.50}$$

where a formula

$$\int_0^\infty e^{-\Gamma t}\cos(\Delta m t)\, dt = \frac{1}{\Gamma}\frac{1}{1+x^2} \quad (x = \frac{\Delta m}{\Gamma}), \tag{9.51}$$

has been used. By a reason stated below we have also set $|\frac{q}{p}| = 1$. In the e^+e^- collider experiments, the ratio r can be measured by observing a ratio

$$\frac{N_{l^+l^+} + N_{l^-l^-}}{N_{l^+l^-}} \quad (\simeq 2r \text{ for small } r \text{ and assuming CP invariance}), \tag{9.52}$$

namely the ratio of the event rates of the same sign leptons to that of the opposite sign leptons. In the collision B^0, $\bar{B^0}$ are pair created, but some of them may experience oscillation, so that there appear a possibility that both of mesons become identical, producing charged leptons as their decay products with the same sign, via processes $b \to c + l^- + \bar{\nu}_l$ or $\bar{b} \to \bar{c} + l^+ + \nu_l$. Historically such experiments first gave an useful information on Δm, which in turn was used to impose the lower bound on m_t.

This mixing due to Δm or the box diagram is a FCNC process of $|\Delta B| = 2$ with B denoting "bottom number". We may think of a CP violating or T violating observable in this $|\Delta B| = 2$ mixing process in mass matrix, sometimes called as "indirect CP violation" (in contrast to "direct CP violation" in the decay amplitude of B meson itself). Namely the event rate difference of the same sign di-lepton events $|N_{l^+l^+} - N_{l^-l^-}|$ will signal a difference $P(B^0 \to \bar{B^0}) - P(\bar{B^0} \to B^0)$, which is a clear signal of CP or T violation. (9.46), (9.47) tells us that the difference is possible only when

$$|\frac{q}{p}|^2 \neq |\frac{p}{q}|^2 \quad \leftrightarrow |\frac{q}{p}| \neq 1. \tag{9.53}$$

Unfortunately, this deviation of $|\frac{q}{p}|$ from 1 is quite small, and to see the CP violating effect is not plausible. The reason is the following. The key ingredient to understand the situation is in (9.38). There we see that if the M_{12} and Γ_{12} have the same phase factor, $|\frac{q}{p}| = 1$, even though $\frac{q}{p}$ itself has a large phase factor (, which produces large CP asymmetries in B decays as we will see later). As we see in Fig. 9.8, the absorptive part Γ_{12} is due to the contributions of u, c quarks. In the hypothetical limit $m_u = m_c$, the orthogonality of the K-M matrix $V_{ub}^* V_{ud} + V_{cb}^* V_{cd} = -V_{tb}^* V_{td}$ implies that M_{12}, handled by t quark, and Γ_{12}, handles by u, c quarks have the same phase factor, leading to $|\frac{q}{p}| = 1$. We thus learn the deviation of $|\frac{q}{p}|$ from 1 is strongly suppressed by a factor $(m_c^2 - m_u^2)/m_t^2$. The difference $m_c^2 - m_u^2$ was compared with m_t^2, since in the limit of $\Gamma_{12}/M_{12} \to 0$, again $|\frac{q}{p}| = 1$ is realized.

9.4 CP Violation in Kobayashi-Maskawa Model

The 3 generation Kobayashi-Maskawa model was proposed to account for the observed CP violation. As we have briefly discussed in the section 9.1, in this model CP violation is closely related with FCNC processes. In other words, the CP violation needs the breaking of the global "horizontal" or generation symmetry $SU(3)$ ($n_g = 3$), which is also needed in FCNC. To see this, suppose $m_u = m_c$. Then, $SU(2)$ sub-group of $SU(3)$ in the up-type quark sector, i.e. the symmetry between u and c quarks becomes an exact symmetry. By use of this $SU(2)$ symmetry, it is always possible to perform a suitable unitary transformation, belonging to the $SU(2)$, so that V_{ud} vanishes, while we can always make the 1st column and the 1st row of the KM matrix real by suitable re-phasing of quark fields $q \to e^{i\phi}\, q$. Then, it is easy to show (see Problem 9.4) that we can perform a re-phasing such that the all matrix elements become real. We learn from this example that if there is a mass degeneracy in either of up- or down-type quark sector, CP violation never happens (except for the possible strong CP violation, which we ignore).

Thus there arises a natural question: what is the (necessary and sufficient) condition for the CP violation in the Kobayashi-Maskawa model? In particular we ask what is the re-phasing invariant quantity which characterizes the CP violation. The importance of the re-phasing invariance should be stressed here. In (4.181), the CP violating phase δ appears in the matrix element of, e.g. V_{tb}, while in (4.183) the

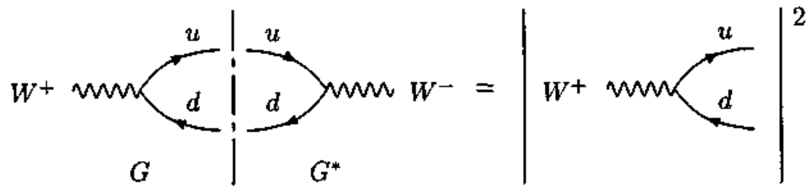

Fig. 9.9

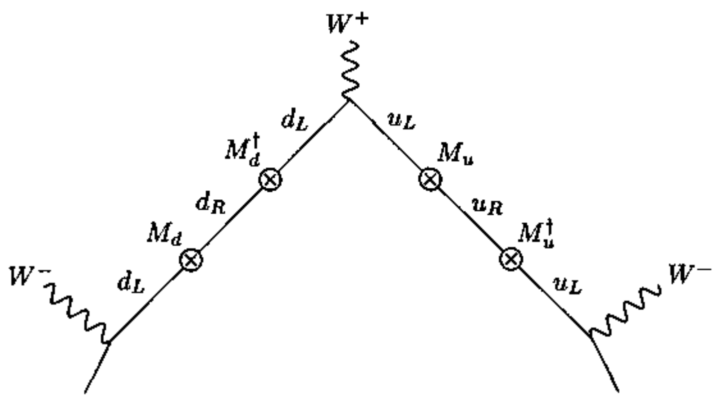

Fig. 9.10

matrix element is real. This is a little puzzling, since if we adopt (4.181) it seems a process with $t \rightarrow b$ transition has CP violating effect, while if we adopt (4.183) there seems to be no CP violation in the process. Actually the CP violating phase of (4.181) can be eliminated by a suitable re-phasing of t, b. As the physics should be the same, irrespectively of the re-phasing, this means it is meaningless to talk about such re-phasing variant quantity V_{tb} to judge whether the CP symmetry is violated or not, and we should seek a re-phasing invariant quantity, which is due to the interference of different KM matrix elements. Let us now come back to the condition for the CP violation. We first note that any observables related to quarks are represented by the cuts of Feynman diagrams with closed fermion loops. In fact if a transition amplitude A is represented by a diagram G, its complex conjugation A^* is represented by a diagram G^* where the directions of the arrows of all fermion lines are reversed, as the complex conjugation implies the charge conjugation. Thus $|A|^2 = AA^*$ is represented by combining G and G^*, namely by the cut of closed fermion loop (see Fig. 9.9 for the example of $W^+ \rightarrow u\bar{d}$ decay).

In the language of weak-eigenstates, the origin of the CP violation is attributed to the complex Yukawa couplings or mass matrices of up- and down-type quarks, M_u, M_d, inserted into the fermion line of the closed loop (see Fig. 9.10).

Actually M_u, M_d appear in the loop diagram in the combination of hermitian matrices $H_u \equiv M_u M_u^*$, $H_d \equiv M_d M_d^*$, as the right handed quarks have no charged weak current interaction. Thus if

$$\mathrm{Im}\ \mathrm{Tr}[P_1(H_d)P_1'(H_u)P_2(H_d)P_2'(H_u)\cdots P_n'(H_u))] \neq 0, \tag{9.54}$$

for some monomials of $H_{u,d}$, $P_1(H_d)$, $P_1'(H_u)$ etc., it is the indication of CP violation. In the case of $n = 1$, we easily find that $\mathrm{Im}\,\mathrm{Tr}[P_1(H_d)P_1'(H_u)] = 0$, as $\{\mathrm{Tr}[P_1(H_d)P_1'(H_u)]\}^* = \mathrm{Tr}[P_1(H_d)P_1'(H_u)]^\dagger = \mathrm{Tr}[P_1'(H_u)P_1(H_d)] = \mathrm{Tr}[P_1(H_d)P_1'(H_u)]$. The first non-zero imaginary part appear at $n = 2$:

$$\mathrm{Im}\ \mathrm{Tr}[H_d H_u H_d^2 H_u^2]. \tag{9.55}$$

It can be shown that there appear no more independent imaginary part (Gronau, Kfir and Loewy, 1986). On the other hand some arithmetic shows that $\mathrm{Im}\ \mathrm{Tr}[H_d H_u H_d^2 H_u^2] \neq 0 \leftrightarrow \det\ [H_u, H_d] \neq 0$. Thus we find the condition for the CP violation is

$$\det\ [M_u M_u^\dagger, M_d M_d^\dagger] \neq 0. \tag{9.56}$$

By use of the fact $H_u = U_{uL}^\dagger\ \mathrm{dia}(m_u^2, m_c^2, m_t^2)\ U_{uL}$ etc. (U_{uL} is the unitary matrix used to diagonalize the up-type quark mass matrix), we can show that the above condition is equivalent to (see Problem 9.5)

$$(m_u^2 - m_c^2)(m_u^2 - m_t^2)(m_c^2 - m_t^2)(m_d^2 - m_s^2)(m_d^2 - m_b^2)(m_s^2 - m_b^2) \times J \neq 0, \tag{9.57}$$

where J is the so-called Jarlskog parameter (Jarlskog, 1985), defined by

$$\begin{aligned} J &\equiv |\mathrm{Im}\ (V_{i\alpha}V_{j\alpha}^* V_{j\beta}V_{i\beta}^*)| = \sin^2\theta_1 \sin\theta_2 \sin\theta_3 \cos\theta_1 \cos\theta_2 \cos\theta_3 \sin\delta \\ &\simeq A^2\lambda^5\eta = O(10^{-6})\ (i,j = u,c,t; \alpha,\beta = d,s,b; i \neq j, \alpha \neq \beta), \end{aligned} \tag{9.58}$$

where A, λ, η are parameters in Wolfenstein's parameterization of KM matrix (see (4.184)). As we will see below, even though there seem to be 9 choices of the combination of i, j, α, β, the imaginary part is unique, up to its sign, corresponding to the fact that in 3 generation model there remains only one CP-violating phase. We take J to be a positive value. As we anticipated, (9.57) clearly shows that if there is a mass degeneracy in either of up- and down-type quark sector, such as $m_u = m_c$ or $m_d = m_s$, CP violation never happens.

Let us also note that J is invariant under re-phasing of quark fields, $u_i \to e^{i\phi_i}\ u_i$ ($u_i = u, c, t$), $d_\alpha \to e^{i\phi_\alpha}\ d_\alpha$ ($d_\alpha = d, s, b$). In fact under the re-phasing, $V_{i\alpha} \to e^{i(-\phi_i+\phi_\alpha)}V_{i\alpha}$ and $J \to e^{i(-\phi_i+\phi_\alpha)}e^{i(\phi_j-\phi_\alpha)}e^{i(-\phi_j+\phi_\beta)}e^{i(\phi_i-\phi_\beta)}\ J = J$. The invariance comes from the fact that we have been considering the closed loop of fermion, where the creation and the annihilation of all quarks happen in pair. The uniqueness of the J parameter is easily understood noting the orthogonality of the unitary matrix V. For instance

$$\mathrm{Im}\ (V_{id}V_{jd}^*V_{js}V_{is}^*) = -\mathrm{Im}\ [V_{id}V_{jd}^*V_{jd}V_{id}^* + V_{id}V_{jd}^*V_{jb}V_{ib}^*] = -\mathrm{Im}\ (V_{id}V_{jd}^*V_{jb}V_{ib}^*). \tag{9.59}$$

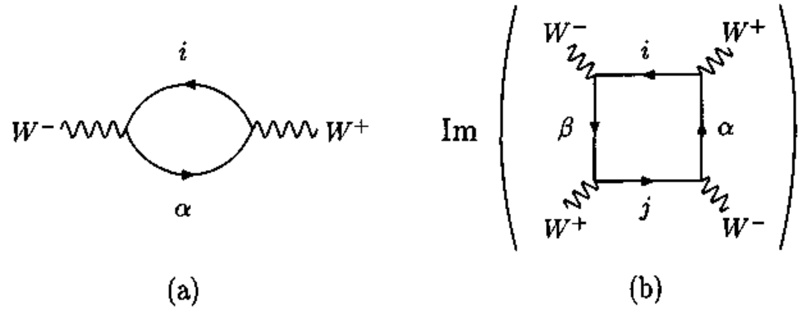

Fig. 9.11

The origin of the J-parameter can be understood diagrammatically. Let us work in mass-eigenstates of quarks. Then the origin of the CP violation is attributed to the CP phase of KM matrix appearing in the vertices of charged currents. We may first think of a diagram with two vertices of charged current where W^+ and W^- are attached, Fig. 9.11(a).

This diagram does not yield an imaginary part, since Im $(V_{i\alpha}V_{i\alpha}^*) = 0$. Next candidate is the diagram given in Fig. 9.11(b), yielding a non-vanishing imaginary part $\mathrm{Im}(V_{i\alpha}V_{j\alpha}^*V_{j\beta}V_{i\beta}^*)$, which is nothing but the J parameter, up to the possible sign difference. We can confirm that there appears no more independent imaginary part, even if additional pairs of W^+ and W^- are attached; for instance, $\mathrm{Im}(V_{i\alpha}V_{j\alpha}^*V_{j\beta}V_{k\beta}^*V_{k\gamma}V_{i\gamma}^*) = -|V_{i\alpha}|^2\mathrm{Im}(V_{j\alpha}^*V_{j\beta}V_{k\beta}^*V_{k\alpha}) - |V_{k\beta}|^2\mathrm{Im}(V_{i\alpha}V_{j\alpha}^*V_{j\beta}V_{i\beta}^*)$ $(i \neq j \neq k \neq i,\ \alpha \neq \beta \neq \gamma \neq \alpha)$.

Suppose the i-th element of α-th and β-th column vectors ($\alpha \neq \beta$) have relative phase factor $V_{i\alpha}/|V_{i\alpha}| = \pm e^{i\phi}$ $(V_{i\beta}/|V_{i\beta}|)$ or $\arg(V_{i\alpha}V_{i\beta}^*) = \phi$, irrespectively of the choice of i. In this case by a suitable re-phasing, e.g. $d_\alpha \to e^{-i\phi}\ d_\alpha$, we can make all of $V_{i\alpha}V_{i\beta}^*$ real, eliminating the relative phase. If suitable re-phasing for each of u_i is further performed, both of α-th and β-th column vectors can be made real, leading to the reality of the third column vector (by a re-phasing) as well, implied by the orthogonality of the column vectors. We thus conclude that, in this case, there should not be any CP violating phenomena. Let us note that if these three complex numbers $V_{i\alpha}V_{i\beta}^*$ $(i = 1, 2, 3)$ are drawn as three vectors in a complex plane, every vectors are parallel or anti-parallel to each other. On the other hand the unitarily of the K-M matrix $\sum_i\ V_{i\alpha}V_{i\beta}^* = 0$ means that the three vectors form a closed triangle, though the two inner angles vanish, or the triangle is a closed one.

This argument suggests that the CP violation is closely related with the inner angles of such triangles, called "unitarity triangles". Namely, three independent orthogonality conditions $\sum_i\ V_{i\alpha}V_{i\beta}^* = 0$ $(\alpha \neq \beta)$ yield three independent unitarity triangles, depending on the choice of the pair (α, β), as shown in Fig. 9.12.

As we will see below, CP asymmetries, such as those discussed in the neutral B meson system, correspond to sin 2θ, with θ being a relevant inner angle of an unitarity triangle. The reason of the necessity of the factor 2 in the argument 2θ

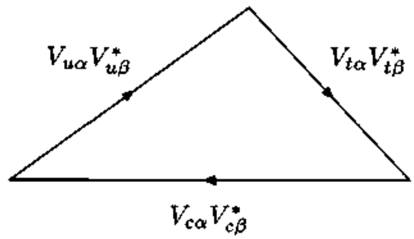

Fig. 9.12

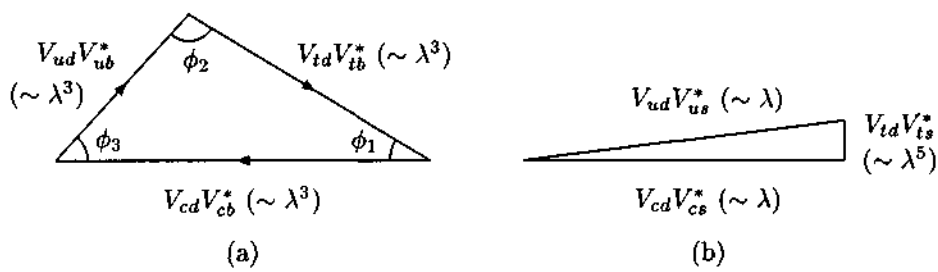

Fig. 9.13

is easily understood; when the triangle is closed the inner angles are either of 0, $\frac{\pi}{2}$ or π, which means sin $2\theta = 0$ for all inner angles. Let us note that now the re-phasing invariance of CP violating observables are manifest; though the each side of the unitarity triangle rotates under the re-phasing, the shape of the triangle is manifestly invariant, so are the inner angles. The shape itself clearly changes depending on the choice of (α, β), while we know every CP violating observable is described by an unique parameter, the Jarlskog parameter J. What happens is, although we have several different triangles depending on (α, β), their area S is known to be unique and just the half of J:

$$J = |\mathrm{Im}\ (V_{i\alpha}V^*_{j\alpha}V_{j\beta}V^*_{i\beta})| = |V_{i\alpha}V^*_{i\beta}||V_{j\alpha}V^*_{j\beta}| \sin\theta_{ij} = 2S, \tag{9.60}$$

where θ_{ij} is the angle between $V_{i\alpha}V^*_{i\beta}$ and $V_{j\alpha}V^*_{j\beta}$ $(\theta_{ij} \leq \pi)$. As long as the area S is unique, the CP symmetry will get large in the system whose triangle has three sizes with comparable lengths, as in the case of B meson system $(\alpha = b, \beta = d)$ we will see later, while if only one of the sizes is very small the inner angles will be near to 0 or $\frac{\pi}{2}$, and the CP asymmetry will get small, as is the case of K meson system $(\alpha = s, \beta = d)$. See Fig. 9.13.

So far our argument has been focused on the CP phase in the KM matrix and the resultant Jarlskog parameter. We have also seen that for CP to be really broken the non-degeneracy of quarks masses in both of the up-type and down type sectors is needed. But, actually we still need an additional condition for the CP violation: the presence of the absorptive part in the relevant transition amplitude (the cut in the Feynman diagram). In fact, in the end of the section 9.3, we have already seen that $\Gamma_{12} = 0$ leads to a vanishing CP violating effect in B meson system. The necessity of the absorptive part is seen from the following illustrative argument. Suppose the transition amplitude of a process A gets contributions form two different Feynman diagrams $A = A_1 + A_2$. We will decompose the each amplitude to the part which possibly contains the CP phase δ due to the weak interaction, and a phase factor indicating the possible presence of the absorptive part due to the on-shell property of the intermediate state: $A_{1,2} = A_{w1,2}\ e^{i\delta_{1,2}}$, $A = A_{w1}\ e^{i\delta_1} + A_{w2}\ e^{i\delta_2}$. The transition amplitude of the process between the CP-conjugated states is written as $A^{(CP)} = A^*_{w1}\ e^{i\delta_1} + A^*_{w2}\ e^{i\delta_2}$. Let us note the phases $\delta_{1,2}$ do not change their sings, since they have dynamical origin and have nothing to do with the KM CP phase. The CP asymmetry proportional to the difference of squared absolute values of A and $A^{(CP)}$ now reads as

$$|A|^2 - |A^{(CP)}|^2 = 4\sin(\delta_1 - \delta_2)\ \mathrm{Im}(A^*_{w1}A_{w2}). \tag{9.61}$$

Here we clearly see that to get CP violation as a physical observable, in addition to the CP phase contributing to $\mathrm{Im}(A^*_{w1}A_{w2})$, the presence of the absorptive parts denoted by $\delta_{1,2}$ is needed. We also see the interference of multiple diagrams, such as $A^*_{w1}A_{w2}$, is inevitable to get the CP violating observable, as suggested by the re-phasing invariance: the CP phase in a single diagram can be always rotated away by a suitable re-phasing of an external quark field. Thus a tree diagram alone never produces any CP violation. We will see below that the possible large CP asymmetries in B system of $\mathcal{O}(10\%)$ is due to the interference of the decay amplitudes of the B meson and the amplitude for $B \leftrightarrow \bar{B}$ mixing, namely the interference between $\Delta B = 1$ and $\Delta B = 2$ amplitudes.

9.5 CP Violation in the Neutral K System

The CP violation had been observed only in the system of neutral kaon system $(K^0, \bar{K^0})$, although the B-factory experiments in KEK and SLAC, recently started, have begun to release the data on the CP asymmetries in the decay processes of neutral B meson system $(B^0, \bar{B^0})$. In fact, historically the first confirmation of the CP violation was made by the detection of the long-lived neutral kaon $K_L \to \pi\pi$. In this section we focus on the CP violation in the kaon system, as it is still playing an important and complimentary roles in the study of CP violation.

As has been already mentioned above, the CP asymmetry in the neutral kaon system turns out to be small. This comes from the fact $|V_{ts}V^*_{td}| \ll |V_{us}V^*_{ud}| \simeq |V_{cs}V^*_{cd}|$ $(|V_{ts}V^*_{td}| = \mathcal{O}(A^2\lambda^5),\ |V_{us}V^*_{ud}| = \mathcal{O}(\lambda))$. The CP asymmetry in the kaon system is thus expected to be $\mathcal{O}(A^2\lambda^4) = \mathcal{O}(10^{-3})$, which has been confirmed experimentally.

Our main concern is the aforementioned $K_L \to \pi\pi$ decay. In the kaon system, in clear contrast to the B meson system, the life times of K_L and K_S are quite different, as suggested by the subscripts L and S, thus making to isolate the K_L state possible. The decay of the K_L state is a clear signature of the CP violation. The state K_L is determined as an eigenstate of the 2×2 Hamiltonian of the $(K^0, \bar{K^0})$ system. If there is no CP violating effect in the Hamiltonian, the eigenstates should be the eigenstates of CP transformation, i.e. $K_L = K_2$ (see (9.7), (9.9)). On the other hand, the K_2 state is odd under the CP transformation, while the final state of the decay, the $\pi\pi$ system, is even under the transformation. This simple argument shows the decay $K_L \to \pi\pi$ never happens, provided the theory is CP invariant. Therefore, its detection should be a clear signature of the CP violation in the theory. This argument also suggests that actually such CP violation decay may have two distinct sources of CP violation, i.e. "indirect CP violation" in the Hamiltonian of the $(K^0, \bar{K^0})$, and the "direct CP violation" in the dacay amplitude $K_2 \to \pi\pi$. We will discuss these issues successively below.

"Indirect CP violation"

The indirect CP violation in the Hamiltonian of the $(K^0, \bar{K^0})$ is due to the imaginary parts of the amplitude of $|\Delta S| = 2$ mixing process $K^0 \leftrightarrow \bar{K^0}$, denoted by M_{12} and Γ_{12}. As the expected CP asymmetry in the kaon system is very small, $\mathcal{O}(10^{-3})$, we expect $\mathrm{Im}M_{12} \ll \mathrm{Re}M_{12}$ etc., and the deviation of the ratio of coefficients of two neutral kaon states q_K/p_K (see (9.36) - (9.38) from unity is small. Thus, writing $p_K/q_K = (1-\bar{\epsilon})/(1+\bar{\epsilon})$, the two eigenstates of the Hamiltonian are re-written as

$$\begin{aligned} |K_L\rangle &= \frac{1}{\sqrt{2(1+|\bar{\epsilon}|^2)}}[(1+\bar{\epsilon})|K^0\rangle + (1-\bar{\epsilon})|\bar{K^0}\rangle] \\ &= \frac{1}{\sqrt{2(1+|\bar{\epsilon}|^2)}}[|K_2\rangle + \bar{\epsilon}|K_1\rangle] \end{aligned} \tag{9.62}$$

$$\begin{aligned} |K_S\rangle &= \frac{1}{\sqrt{2(1+|\bar{\epsilon}|^2)}}[(1+\bar{\epsilon})|K^0\rangle - (1-\bar{\epsilon})|\bar{K^0}\rangle] \\ &= \frac{1}{\sqrt{2(1+|\bar{\epsilon}|^2)}}[|K_1\rangle + \bar{\epsilon}|K_2\rangle]. \end{aligned} \tag{9.63}$$

We learn that non-vanishing $\bar{\epsilon}$ indicates K_L, K_S are not pure eigenstates of CP, implying CP symmetry is violated in the Hamiltonian. As $\mathrm{Im}(V_{cs}V_{cd}^*) = -\mathrm{Im}(V_{ts}V_{td}^*)$ (see (4.183) and (4.184)) and the contributions of the intermediate quarks in the box daigram behave as $E(x_c)/E(x_t) \sim m_c^2/m_t^2 \ll 1$ (see (9.13)), we conclude $\mathrm{Im}\Gamma_{12} \ll \mathrm{Im}M_{12}$. The experimental values tell us $\Delta\Gamma_K = \Gamma_L - \Gamma_S (\simeq -\Gamma_S) = 2\Gamma_{12} \simeq -2\Delta m_K = -4M_{12}$, i.e. $\Gamma_{12} \simeq -2M_{12}$. We thus get a relation from (9.38)

$$\bar{\epsilon} \simeq \frac{e^{\frac{\pi}{4}i}}{\sqrt{2}} \left(\frac{\mathrm{Im}M_{12}}{\Delta m_K}\right). \tag{9.64}$$

Let us note that $\bar{\epsilon}$ cannot be physical observable, since it is re-phasing variant: under the $U(1)$ global transformation concerning strangeness S, $s \to e^{i\phi}s$ or $K^0 \to$

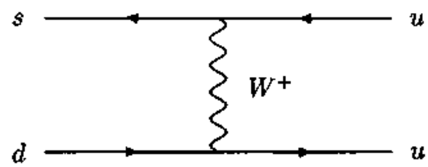

Fig. 9.14

$e^{-i\phi}K^0$, $\bar{K^0} \to e^{i\phi}\bar{K^0}$, $\frac{q_K}{p_K}(=\frac{1-\bar{\epsilon}}{1+\bar{\epsilon}}) \to e^{-2i\phi}\frac{q_K}{p_K}$. Or, for infinitesimal transformation $|\phi| \ll 1$, $\bar{\epsilon} \to \bar{\epsilon} - i\phi$. Thus, though $\mathrm{Re}\bar{\epsilon} \simeq \frac{1}{2}(1 - |\frac{1-\bar{\epsilon}}{1+\bar{\epsilon}}|)$ is re-phasing invariant, $\mathrm{Im}\bar{\epsilon}$ is not re-phasing invariant.

The re-phasing invariant $\mathrm{Re}\bar{\epsilon}$ has been measured by the observation of the "charge asymmetry"

$$\begin{aligned}\delta_L &\equiv \frac{\Gamma(K_L \to \pi^- l^+ \nu_l) - \Gamma(K_L \to \pi^+ l^- \bar{\nu}_l)}{\Gamma(K_L \to \pi^- l^+ \nu_l) + \Gamma(K_L \to \pi^+ l^- \bar{\nu}_l)} = \frac{|p_K|^2 - |q_K|^2}{|p_K|^2 + |q_K|^2} \\ &\simeq 1 - |\frac{q_K}{p_K}| \simeq 2\mathrm{Re}\bar{\epsilon} = (3.30 \pm 0.12) \times 10^{-3}, \end{aligned} \tag{9.65}$$

which clearly vanishes if CP is conserved, as $\pi^- l^+ \nu_l$ and $\pi^+ l^- \bar{\nu}_l$ are CP conjugate states of one another. As we expected, the observed CP asymmetry is of $\mathcal{O}(10^{-3})$. This charge asymmetry is equivalent to the non-orthogonality of K_L, K_S: $\delta_L = \langle K_L | K_S \rangle$.

Including "direct CP violation"

CP violating effect may "directly" appear in the amplitude of $K^0, \bar{K^0} \to \pi\pi$ themselves. As a pion has an isospin $I = 1$, the final state $\pi\pi$ has either $I = 0$ or 2 ($I = 1$ contradicts with Bose statistics of scalar fields). Write the amplitudes of the decay to these two possible isospin states as

$$A(K^0 \to \pi\pi(I)) = e^{i\delta_I}\, A_I \quad (I = 0, 2), \tag{9.66}$$

where A_I are due to the weak interaction, including possible CP phase, while δ_I denote the absorptive parts coming from the strong (final state) interactions. The corresponding amplitudes for $\bar{K^0}$ decays are obtained by replacing A_I by $-A_I^*$, without changing the signs of δ_I. The A_0 gets a remarkable contribution from the "penguin" contribution with gluon-exchange (see Fig. 9.6), which is expected to be responsible for the relative enhancement of the $I = 0$ amplitude compared with $I = 2$ amplitude in the $K \to \pi\pi$ decays, $\frac{\mathrm{Re}A_0}{\mathrm{Re}A_2} \simeq 22$, so called $\Delta I = \frac{1}{2}$ rule, while the A_2 gets the main contribution from the ordinary charged current process shown in Fig. 9.14, though its CP violating imaginary part may be provided by "electroweak penguin" diagram, which is obtained by replacing the gluon by γ or Z in the penguin diagram.

The final state $\pi\pi$ is either $\pi^+\pi^-$ or $\pi^0\pi^0$. Write the corresponding measures of

CP violation as

$$\eta_{+-} \equiv \frac{A(K_L \to \pi^+\pi^-)}{A(K_S \to \pi^+\pi^-)} \equiv \epsilon + \epsilon' \tag{9.67}$$

$$\eta_{00} \equiv \frac{A(K_L \to \pi^0\pi^0)}{A(K_S \to \pi^0\pi^0)} \equiv \epsilon - 2\epsilon', \tag{9.68}$$

in terms of two parameters ϵ and ϵ', which denote the part common for two kinds of decay and the difference of the amplitudes of the decays, respectively. By use of a relation

$$|\pi^+\pi^-\rangle = \sqrt{\frac{2}{3}}\,|\pi\pi(I=0)\rangle + \sqrt{\frac{1}{3}}\,|\pi\pi(I=2)\rangle \tag{9.69}$$

$$|\pi^0\pi^0\rangle = -\sqrt{\frac{1}{3}}\,|\pi\pi(I=0)\rangle + \sqrt{\frac{2}{3}}\,|\pi\pi(I=2)\rangle, \tag{9.70}$$

and $A(K_L \to \pi\pi)/A(K_S \to \pi\pi) \simeq (\bar{\epsilon}A(K_1 \to \pi\pi) + A(K_2 \to \pi\pi))/A(K_1 \to \pi\pi) = \bar{\epsilon} + (A(K_2 \to \pi\pi)/A(K_1 \to \pi\pi))$, we easily get the following approximate relation

$$\epsilon \simeq \bar{\epsilon} + i\,(\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}) \tag{9.71}$$

$$\epsilon' \simeq \frac{i}{\sqrt{2}}e^{i(\delta_2-\delta_1)}\mathrm{Im}(\frac{A_2}{A_0}) \simeq \frac{i}{\sqrt{2}}e^{i(\delta_2-\delta_1)}(\frac{\mathrm{Re}A_2}{\mathrm{Re}A_0})(\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}), \tag{9.72}$$

where we have used the facts A_2 is sufficiently smaller than A_0 in both in their real part ($\Delta I = 1/2$ rule) and in their imaginary part (electromagnetic penguin contribution is suppressed compared to ordinary penguin contribution by the difference of gauge couplings). As K_L and π are physical states, the decay amplitudes, and therefore ϵ, ϵ', should be re-phasing invariant. In fact, neglecting A_2,

$$\epsilon = \frac{A(K_L \to \pi\pi)}{A(K_S \to \pi\pi)} = \frac{1 - \frac{q_K}{p_K}\frac{A_0^*}{A_0}}{1 + \frac{q_K}{p_K}\frac{A_0^*}{A_0}} \simeq \frac{1}{2}[1 - \frac{q_K}{p_K}\frac{A_0^*}{A_0}]. \tag{9.73}$$

As $\frac{q_K}{p_K} \to e^{-2i\phi}\frac{q_K}{p_K}$, and $A_0 \to e^{-i\phi}A_0$ under the strangeness $U(1)$ transformation, the combination $\frac{q_K}{p_K}\frac{A_0^*}{A_0}$ is re-phasing invariant. The re-phasing invariance of ϵ' is trivial.

In the KM model, $\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}$ is less important compared with $\bar{\epsilon}$, and $\epsilon \simeq \bar{\epsilon}$. Thus in terms of the coefficient function $E(x)$ obtained from the box diagram,

$$\epsilon \simeq -e^{\frac{\pi}{4}i}\frac{\alpha G_F}{24\pi\sin^2\theta_W}\frac{BM_K f_K^2}{\Delta m_K}\cdot \mathrm{Im}\{(V_{ts}^* V_{td})^2\}\cdot E(\frac{m_t^2}{M_W^2}), \tag{9.74}$$

though additional factor coming from perturbative QCD correction of $\mathcal{O}(1)$ should also be multiplied. The experimental value (Particle Data Group, 2002)

$$|\epsilon| = (2.282 \pm 0.017) \times 10^{-3}, \tag{9.75}$$

is useful to restrict the shape of the unitarity triangle, or on $(1-\rho)\eta$ (in terms of Wolfenstein's parameterization) from the factor $\mathrm{Im}\{(V_{ts}^{*}V_{td})^2\}$, since we now know $m_t \simeq 175\mathrm{GeV}$. The prediction of another CP violating variable ϵ' is less reliable, since we have two competing terms $\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2}$ and $\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}$. Though $|\mathrm{Im}A_2| \ll |\mathrm{Im}A_0|$, the $\Delta I = 1/2$ rule relatively enhances the importance of $\mathrm{Im}A_2$, which gets a contribution from the electro-weak penguin diagram. As the coefficient function of the FCNC $\bar{s}dZ$ vertex, $\Gamma^Z(\frac{m_t^2}{M_W^2})$ in (9.20) grows roughly in proportion to m_t^2 for larger m_t, the non-decoupling contribution of the intermediate heavy t quark tends to cancel the effect of $\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}$ coming from ordinary penguin diagram, thus making the prediction of ϵ' rather smaller than naively expected, i.e. $|\frac{\epsilon'}{\epsilon}| = \mathcal{O}(10^{-4})$ to $\mathcal{O}(10^{-3})$, while the present experimental data (Particle Data Group, 2002),

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right) \approx \frac{\epsilon'}{\epsilon} = (1.8 \pm 0.4) \times 10^{-3}, \tag{9.76}$$

seem to support this expectation. In accordance with our general argument, the CP violating observables ϵ, ϵ' result from the interference effects of different amplitudes, e.g. the interference of A_0 and A_2 in the case of ϵ'. We may summarize this section by classifying the CP violating observables depending on where they have their origin, i.e. in the $\Delta S = 2$ "indirect" CP violation in the mass matrix, or in the $\Delta S = 1$ "direct" CP violation in the decay amplitudes or in the interference of these two types:

· pure $\Delta S = 2$ CP violation: $\mathrm{Re}\ \epsilon$ (δ_L)
· interference between $\Delta S = 2$ and $\Delta S = 1$: $\mathrm{Im}\ \epsilon$
· pure $\Delta S = 1$ CP violation: ϵ'.

9.6 CP Violation in the Neutral B System

In the KM model, as long as there are three independent unitarity triangles depending on the choice of (α, β) with $\alpha, \beta = d, s, b$, there should be CP violating phenomena, not only in the K system ($\alpha = d, \beta = s$), but also in the neutral B meson systems, B_d ($\alpha = d, \beta = b$) and B_s ($\alpha = s, \beta = b$). Hence, the observations of the CP violating events, in the rates predicted by the KM model will be very useful for the final confirmation of the model, as the model for explaining CP violation, or by chance we may even be able to find some deviation from the prediction, which then will be an indication of some "new physics". In addition, as was discussed in section 9.4, we expect large CP asymmetries, of $\mathcal{O}(10\%)$, in the B_d system, stemming from the fact that the three sizes of the unitarity triangle for $(\alpha = d, \beta = b)$ are all of comparable lengths, $\mathcal{O}(A\lambda^3)$. These are main motivations for currently on-going B-factory experiments at KEK and SLAC. In this section we mainly discuss the B_d system, and we will simply use the abbreviation B^0 for B_d, unless otherwise mentioned.

Although the origin of the CP violation in the B system in KM model is the same as in the K system, i.e. the violation of flavor symmetry and the CP violating J-parameter, there are a few characteristic features of the B system listed below, which requires some sophisticated observational method of the CP asymmetries in the system, a bit different from that in the K system:

a. In the B system, since $\Delta\Gamma \ll \Delta m \sim \Gamma$ ($\Gamma_{12}/M_{12} \sim m_b^2/m_t^2$), the two eigenstates of the Hamiltonin, B_1, B_2 defined in (9.36), (9.37), have almost the same life time. Thus, in contrast to the case of kaon system, we cannot extract pure B_2 state, corresponding to K_L: the observation of CP violation by use of the decays of B_1 or B_2 is practically impossible, unless one of these states has been prepared as an initial state.

b. In the collision experiments such as those in the B-factories, the initial states should be either B^0 or $\bar{B^0}$, the (coherent) admixtures of B_1, B_2 states, and we expect the flavor oscillation $B^0 \leftrightarrow \bar{B^0}$, as was discussed in 9.3, which is similar to the neutrino oscillations, except the B mesons decay as the time goes by with the almost equal life time $\tau = 1/\Gamma$. If the oscillation in the duration of the lifetime is too rapid, i.e. if $x \equiv \Delta m/\Gamma = \Delta m\ \tau \gg 1$, we in practice cannot distinguish B^0 and $\bar{B^0}$ states, and the CP asymmetries, sensitive to the difference of these two states, will be washed out, no matter which initial state we prepare. Fortunately, the obtained value of x from the observation of the same sign leptons in e^+e^- collisions (see (9.50)) is in a suitable range in order to avoid such washing out, $x \sim 0.7$.

c. Although we generally expect the CP asymmetry in the B system is quite large, the pure $\Delta B = 2$ CP asymmetry, $\propto P(B^0 \to \bar{B^0}) - P(\bar{B^0} \to B^0)$, is much smaller than the naive expectation, being suppressed by $|\frac{q}{p}| - 1 \sim \frac{m_b^2}{m_t^2}$ (see the argument below (9.53)). We also note the rates of "direct" CP violation in the pure $\Delta B = 1$ decay amplitudes cannot be predicted with accuracy: the (partial) decay rate asymmetries $\propto \Gamma(B^0 \to f) - \Gamma(\bar{B^0} \to \bar{f})$ ($f, \bar{f}$: distinct final states) need the interference among different amplitudes, which have absorptive parts with different phases, whose estimation is not easy especially for those caused by strong interactions.

d. Thus the remaining candidate to see the sizable CP asymmetries with reliable predictions, is the CP asymmetries caused by the interference effect between $\Delta B = 2$ and $\Delta B = 1$ amplitudes, corresponding to Im ϵ in the K system. In fact, we will see below the CP asymmetry is due to

$$\mathrm{Im}\,\left(\frac{q}{p}\frac{A(\bar{B^0} \to f)}{A(B^0 \to f)}\right), \tag{9.77}$$

which corresponds to $-2\mathrm{Im}\,\epsilon$ in K system (see (9.73)). Such type of CP asymmetries is known to be of $\mathcal{O}(10\%)$, as was first pointed out by Sanda and collaborators (Bigi and Sanda, 2000). Another nice thing of this type of CP asymmetry is the possible phase due to strong interaction exactly cancelled out, as long as we choose a final state f such that the decay amplitude A is dominated by one Feynman diagram, as in the case of $f = \psi K_S$. In this type of CP violation $B^0 \leftrightarrow \bar{B^0}$

oscillation obviously plays a central role. In fact, the CP asymmetry is caused by the interference of two distinct amplitudes, $B^0(\bar{B^0}) \to f$ and $B^0(\bar{B^0}) \to \bar{B^0}(B^0) \to f$, realized by the oscillation. (We may rewrite the CP violating factor Im $(\frac{q}{p}\frac{A(\bar{B^0}\to f)}{A(B^0\to f)})$ as $\frac{1}{|A(B^0\to f)|^2}$ Im $(\frac{q}{p}\ A(\bar{B^0} \to f)\ A(B^0 \to f)^*)$.) Thus we need a non-vanishing Δm for the oscillation, and the observed value of $x \simeq 0.7$ is in an ideal range.

e. Although the CP asymmetry of this type is expected to be quite large, this in turn means the lengths of the three sizes of the relevant unitarity triangle are tiny, $\mathcal{O}(A\lambda^3)$. Namely the branching ratios of the decay processes to observe the CP asymmetries are rather small, e.g. $Br(B \to \psi K_S) = \mathcal{O}(10^{-4})$. Thus, even though the CP asymmetries are sizable, we still need high luminosity for the collision experiments. Let us note that in the K system, $K_S \to \pi\pi$ was the main decay mode. Of course, the large CP asymmetries will help a lot in the statistical significance of the signal: needed luminosity is proportional to $1/A^2$, instead of $1/A$, with A being a generic CP asymmetry.

We are now ready to discuss the sizable CP asymmetry caused by the $B^0 \leftrightarrow \bar{B^0}$ oscillation. For instance in e^+e^- collision experiments, $e^+e^- \to B^0\bar{B^0},\ B^+B^-$, we a priori do not know which one of two neutral mesons is B^0, which makes the observation of CP asymmetry difficult. We start with the idealized situation, where we know that at time 0 B^0 or $\bar{B^0}$ was produced, and denote their state at time t as $B^0(t)$ or $\bar{B^0}(t)$ with $B^0(0) = B^0$ or $\bar{B^0}(0) = \bar{B^0}$, which are linear combinations of B^0 and $\bar{B^0}$, as shown in (9.46) and (9.47). Let us consider a CP asymmetry, which is the difference of the probabilities, or event rates, for the state starting as B^0 or $\bar{B^0}$ to decay into the final state f or $\bar{f}$ at time t, divided by the sum of the probabilities:

$$A(t) \equiv \frac{P(B^0(t) \to f) - P(\bar{B^0}(t) \to \bar{f})}{P(B^0(t) \to f) + P(\bar{B^0}(t) \to \bar{f})}, \tag{9.78}$$

where

$$P(B^0(t) \to f) \quad \propto \quad |g_+(t)A(B^0 \to f) - \frac{q}{p}g_-(t)A(\bar{B^0} \to f)|^2, \tag{9.79}$$

$$P(\bar{B^0}(t) \to \bar{f}) \quad \propto \quad |g_+(t)A(\bar{B^0} \to \bar{f}) - \frac{p}{q}g_-(t)A(B^0 \to \bar{f})|^2. \tag{9.80}$$

As we have mentioned above, we can safely assume $|\frac{q}{p}| = 1$. We also take a CP self-conjugate state as the final state, $\bar{f} = \pm f$, which has a merit to allow the both processes $B^0 \to f$ and $\bar{B^0} \to f$ possible, and to eliminate the uncertainty in the phases due to the strong interaction, "strong phases", in the ratio $\frac{A(\bar{B^0}\to f)}{A(B^0\to f)}$: As the strong interaction preserves CP, generally the strong phases of $A(B^0 \to f)$ and $A(\bar{B^0} \to \bar{f})$ are exactly the same, which means for the case of $\bar{f} = f$ the ratio is free from the strong phase. The typical examples are $f = \psi K_S,\ \pi\pi$. Under these assumption the time-dependent asymmetry $A(t)$ may be written as

$$\frac{\cos(\Delta mt)[\Gamma(B^0 \to f) - \Gamma(\bar{B^0} \to f)] + 2\sin(\Delta mt)\Gamma(B^0 \to f)\mathrm{Im}\,(\frac{q}{p}\frac{A(\bar{B^0}\to f)}{A(B^0\to f)})}{(\Gamma(B^0 \to f) + \Gamma(\bar{B^0} \to f))}, \tag{9.81}$$

where, the partial decay widths are proportional to the squared-transition- amplitudes:

$$\Gamma(B^0 \to f) \propto |A(B^0 \to f)|^2, \;\; \Gamma(\bar{B^0} \to f) \propto |A(\bar{B^0} \to f)|^2. \tag{9.82}$$

In the expression for the time-dependent CP asymmetry $A(t)$, the term accompanied by $\cos(\Delta mt)$ does not go away at $t = 0$, and is the indication of the direct CP violation in the $\Delta B = 1$ decay amplitude, which does not need the help of the flavor oscillation, while the term accompanied by $\sin(\Delta mt)$ and Im $(\frac{q}{p}\frac{A(\bar{B^0}\to f)}{A(B^0\to f)})$ vanishes at $t = 0$, and indicates the CP violation due to the interference between $\Delta B = 2$ and $\Delta B = 1$ amplitudes, with large CP asymmetry, we are seeking. As we have discussed above, we are interested in the process where the decay into f is dominated by one Feynman diagram. In that case the direct CP violation is negligible, i.e. $\Gamma(B^0 \to f) \simeq \Gamma(\bar{B^0} \to f)$. In this way, the formula for the CP asymmetry greatly simplified into

$$A(t) = \sin(\Delta mt) \cdot \mathrm{Im}\,(\frac{q}{p}\frac{A(\bar{B^0} \to f)}{A(B^0 \to f)}). \tag{9.83}$$

It seems the time-integrated or time-averaged CP asymmetry vanishes, as the time-average of $\sin(\Delta mt)$ just goes away. This is the case provided $\Gamma = 0$, as in the CP asymmetries in neutrino oscillations, The finite lifetime of B mesons, however, makes the smearing of the CP asymmetry due to the oscillation incomplete. From a relation,

$$(\int_0^\infty e^{-\Gamma t} \sin\,(\Delta mt)\, dt)/(\int_0^\infty e^{-\Gamma t}\, dt) = \frac{x}{1+x^2}, \tag{9.84}$$

we get the time-integrated asymmetry,

$$\begin{aligned} A &\equiv \frac{\int_0^\infty dt\, [P(B^0(t) \to f) - P(\bar{B^0}(t) \to \bar{f})]}{\int_0^\infty dt\, [P(B^0(t) \to f) + P(\bar{B^0}(t) \to \bar{f})]} \\ &= \frac{x}{1+x^2}\,\mathrm{Im}\,(\frac{q}{p}\frac{A(\bar{B^0} \to f)}{A(B^0 \to f)}), \end{aligned} \tag{9.85}$$

where the smearing factor $\frac{x}{1+x^2} \simeq 0.5$ is fortunately not small. Write the time-independent CP asymmetry factor as $a(f) \equiv \mathrm{Im}\,(\frac{q}{p}\frac{A(\bar{B^0}\to f)}{A(B^0\to f)})$. As the three sizes of the unitarity triangle are of comparable lengths, and the coefficient function representing the contributions of $u_i = u, c, t$ to $\Delta B = 2$ box diagram grows roughly as $\sim m_i^2/M_W^2$ the M_{12}, Γ_{12} stemming form the box diagram are dominated by the t quark contributions. Thus $\frac{q}{p} \simeq \sqrt{\frac{M_{12}^*}{M_{12}}}$ is written in terms of KM matrix elements related to t quark, $(V_{ts}V_{td}^*)^2$. Thus for the most promising channel to observe the CP asymmetry in the B-factory experiments,

$$a(\pi K_S) \simeq \mathrm{Im}(\frac{(V_{tb}V_{td}^*)^2}{|V_{tb}V_{td}^*|^2}) = -\sin 2\phi_1, \tag{9.86}$$

where $\frac{V_{td}V_{tb}^*}{|V_{td}V_{tb}^*|} = e^{i(\phi_1 - \pi)}$ was used (see Fig. 9.13(a)). As we expected, the CP asymmetry is described by the sin of an inner angle of the unitarity triangle in (b,d) sector. In terms of Wolfenstein's parameters ρ and η of $\mathcal{O}(1)$ (see (4.184)),

$$a(\pi K_S) \simeq -\frac{2(1-\rho)\eta}{(1-\rho)^2 + \eta^2}, \tag{9.87}$$

and the $|a(\pi K_S)|$ can be of order 10%. This large CP asymmetry has already been observed at B-factories: Belle and BaBar experiments have reported $|a(\pi K_S)| \simeq$ 70%.

One may wonder why the expression above is not re-phasing invariant: the phase of $(V_{ts}V_{td}^*)^2$ changes by the re-phasing. Actually when the factor $\frac{A(\bar{B}^0\to\pi K_S)}{A(B^0\to\pi K_S)} \simeq \frac{(V_{cb}^*V_{cs})p_K^*}{V_{cb}V_{cs}^*q_K^*} \simeq \frac{(V_{cb}^*V_{cs})^2(V_{cs}^*V_{cd})^2}{|V_{cb}^*V_{cs}|^2|V_{cs}^*V_{cd}|^2}$ is taken into account, we realize that the whole expression is re-phasing invariant, as it should be. Let us note that $V_{cb}^*V_{cs}$ comes from the decay $\bar{b} \to \bar{c}c\bar{s}$ due to the ordinary charged current process, while $p_K^*/q_K^* \simeq (V_{cs}^*V_{cd})/(V_{cs}V_{cd}^*)$ reflects that K^0 and $\bar{K}^0$ are contained in K_S by the fractions p_K and q_K. As the matter of fact, however, $V_{cb}^*V_{cs}$ is almost real and the CP violating phase appearing in $p_K^*/q_K^* \simeq 1 + 2\bar{\epsilon}^*$ is suppressed by $|\bar{\epsilon}| = \mathcal{O}(10^{-3})$. That is why the CP asymmetry is dominated by the phase of $\frac{V_{td}V_{tb}^*}{|V_{td}V_{tb}^*|}$.

So far we have assumed that we can prepare a neutral B meson with definite flavor, i.e. B^0 or $\bar{B}^0$, at $t = 0$. Such identification is crucial when we discuss CP asymmetry, since the interchange $B^0 \leftrightarrow \bar{B}^0$ at $t = 0$ will change the sign of CP asymmetry. Unfortunately such identification of flavor is not possible in a realistic situation of e^+e^- experiment, such as the B-factory experiments Belle and BaBar, since B^0 and $\bar{B}^0$ are produced in a pair. In addition the presence of $B^0 \leftrightarrow \bar{B}^0$ oscillation makes the identification difficult. One intriguing idea is to tag the flavor of one of the mesons to be, e.g. $\bar{B}^0$, by the observation of its decay products at some time t. As B mesons are produced at the collider experiment through $e^+e^- \to \Upsilon(4S) \to B^0\bar{B}^0$, the relative orbital angular momentum l of two mesons is 1, i.e. p-wave, whose wave function is antisymmetric under the exchange of two mesons. Thus the remaining meson at time t should have opposite flavor, e.g. B^0, even though two mesons may have the same flavor at different times as the result of the oscillation. If we regard the time of the tagging t as a sort of initial time we are able to repeat the argument extended above for the idealized situation.

Let us illustrate the idea more concretely by explicit calculations. At the initial time when B mesons are produced via the decay of Υ, the state of two mesons with momenta k and k' is described by

$$\frac{1}{\sqrt{2}}[|B^0(k)\rangle|\bar{B}^0(k')\rangle + c|B^0(k')\rangle|\bar{B}^0(k)\rangle], \tag{9.88}$$

where $c = (-1)^l$. Though in the on-going B-factory experiment $l = 1,\ c = -1$ as explained, $c = 1$ is possible, e.g. in the process $\Upsilon(5S) \to B^0\bar{B}^{0*} \to B^0\bar{B}^0\gamma$, where $l = 0$ is possible. Let f_a a final state, which both of B^0 and $\bar{B}^0$ can decay into, i.e. $\bar{f}_a = f_a$, and f_b a final state, which only $\bar{B}^0$ can decay into. Namely f_b can be utilized for the tagging of $\bar{B}^0$. Let $P(f_a, f_b; t, t')_c dt dt'$ be the probability for the mesons with momenta k and k' to decay into f_a and f_b during the time intervals $(t, t + dt)$ and $(t', t' + dt')$, respectively. Then,

$$\begin{aligned} & P(f_a, f_b; t, t')_{\mp} \\ \propto\ & |\frac{1}{\sqrt{2}}[< f_a|B^0(t) >< f_b|\bar{B}^0(t') > +c < f_b|B^0(t') >< f_b|\bar{B}^0(t) >]|^2 \\ \propto\ & \Gamma(B^0 \to f_a)\Gamma(\bar{B}^0 \to f_b)e^{-\Gamma t_+} \\ \cdot\ & \{1 - \text{Im}\ (\frac{q}{p}\frac{A(\bar{B}^0 \to f_a)}{A(B^0 \to f_a)})\ \sin(\Delta m t_{\mp})\}, \end{aligned} \tag{9.89}$$

where $t_{\pm} \equiv t \pm t'$ $(t_+ \geq 0,\ -t_+ < t_- < t_+)$. Similarly,

$$\begin{aligned} & P(f_a, \bar{f}_b; t, t')_{\mp} \\ \propto\ & \Gamma(B^0 \to f_a)\Gamma(\bar{B}^0 \to f_b)e^{-\Gamma t_+} \\ \cdot\ & \{1 + \text{Im}\ (\frac{q}{p}\frac{A(\bar{B}^0 \to f_a)}{A(B^0 \to' f_a)})\ \sin(\Delta m t_{\mp})\}, \end{aligned} \tag{9.90}$$

where a good approximations $\Gamma(B^0 \to f_a) \simeq \Gamma(\bar{B}^0 \to f_a)$, and $\Gamma(B^0 \to \bar{f}_b) \simeq \Gamma(\bar{B}^0 \to f_b)$ have been used, which are valid for decay processes dominated by tree level diagrams. The subscript $\mp$ correspond to $c = \mp 1$. At the time t' of tagging of the meson with momentum k', another meson is known to have different flavor, and if we regard t_- as the time t in the ideal situation it is expected that we have a CP asymmetry with the same time dependence as what we got above for the idealizes case. In fact, for the case of our main interest $c = -1$, after the integration over t_+

$$\int_{|t_-|}^{\infty} P(f_a, f_b; t, t')_-\ dt_+ \propto e^{-\Gamma|t_-|}\{1 - \text{Im}\ (\frac{q}{p}\frac{A(\bar{B}^0 \to f_a)}{A(B^0 \to f_a)})\ \sin(\Delta m t_-)\}, \tag{9.91}$$

$$\int_{|t_-|}^{\infty} P(f_a, \bar{f}_b; t, t')_-\ dt_+ \propto e^{-\Gamma|t_-|}\{1 + \text{Im}\ (\frac{q}{p}\frac{A(\bar{B}^0 \to f_a)}{A(B^0 \to f_a)})\ \sin(\Delta m t_-)\},. \tag{9.92}$$

As we expected if we replace t_- by t we recover the previous results. One remarkable difference from the previous case is that now t_- is not positive-definite, but may be negative $(t < t')$. Thus if we further integrate the probabilities over t_-, the term proportional to the CP asymmetry $\text{Im}\ (\frac{q}{p}\frac{A(\bar{B}^0 \to f_a)}{A(B^0 \to f_a)})$ will disappear. Thus we need to identify the decay times t and t'. This is why the on-going B-factory experiments are performed by asymmetric collider where the energies of e^+ and e^- are different and therefore the center of mass system is moving. Thus the difference of t and t' is measurable by identifying the corresponding decay points. If we denote the number of events for the two mesons to decay into f_a and f_b at time t and t' with $t > t'$, etc. by $N(f_a, f_b)_{t>t'}$ etc., the time-integrated CP asymmetry $A = \frac{x}{1+x^2}\, a(f_a)$ is given as

$$A = \frac{N(f_a, f_b)_{t>t'} - N(f_a, \bar{f}_b)_{t>t'} - N(f_a, f_b)_{t<t'} + N(f_a, \bar{f}_b)_{t<t'}}{N(f_a, f_b)_{t>t'} + N(f_a, \bar{f}_b)_{t>t'} + N(f_a, f_b)_{t<t'} + N(f_a, \bar{f}_b)_{t<t'}}. \tag{9.93}$$

Problems

9.1 Consider the Yukawa couplings of two Higgs doublets ϕ_1, ϕ_2, given in (9.3). Show that a linear combination of their neutral components $\frac{1}{\sqrt{v_1^2+v_2^2}}(v_1\varphi_1^0 + v_2\varphi_2^0)$ ($v_{1,2}^0 = \langle\varphi_{1,2}^0\rangle$) has flavor-diagonal Yukawa coupling (no FCNC).

9.2 Utilize the effective FCNC γ-vertex given in (9.21) and (9.23), (9.24) to get the effective FCNC gluon-vertex, i.e. the contribution of "penguin" diagram. The comment below (9.24), concerning the Q-dependence of the effective γ-vertex may be helpful.

9.3 Solve the time-evolution equation (9.42) to get the solution (9.44), (9.45).

9.4 Prove that when $m_u = m_c$, for instance, all the elements of the KM (Kobayashi-Maskawa) matrix can be made real, by use of suitable re-phasing of quark fields.

9.5 Verify that the condition (9.56) is equivalent to (9.57).

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Appendix A

Notation and Useful Relations

A.1 Four vectors

Throughout this book, we use the natural unit $\hbar(\equiv h/2\pi) = c = 1$, where h is the Planck constant and c is the light velocity. We follow the convention of Bjorken and Drell (Bjorken and Drell, 1964).

Space-time coordinates $x \equiv (t, x, y, z) = (t, \vec{x})$ are described by the contravariant 4-vector

$$x^\mu \equiv (x^0, x^1, x^2, x^3) = (t, x, y, z) . \tag{A.1}$$

The metric tensor is defined as

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} , \tag{A.2}$$

and produces the covariant vector

$$x_\mu = g_{\mu\nu}x^\nu = (x_0, x_1, x_2, x_3) = (t, -x, -y, -z) \tag{A.3}$$

The repeated indices are summed unless otherwise specified. Then the scalar product of space-time points is given as

$$x^2 \equiv x^\mu x_\mu = g_{\mu\nu}x^\mu x^\nu = g^{\mu\nu}x_\mu x_\nu = t^2 - \vec{x}^2 . \tag{A.4}$$

The four-momenta are similarly given as

$$p^\mu = (p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z) . \tag{A.5}$$

The four-gradient is defined as $\partial_\mu \equiv \partial/\partial x^\mu$ or $\partial^\mu \equiv \partial/\partial x_\mu$. Then the momentaum operator in the coordinate representation is given as

$$p^\mu = i\partial^\mu = \left(i\frac{\partial}{\partial t}, -i\nabla\right) . \tag{A.6}$$

Thus,

$$p^\mu p_\mu = -\partial^\mu \partial_\mu = -\frac{\partial^2}{\partial t^2} + \nabla^2 \equiv -\Box\,, \tag{A.7}$$

where ∇^2 and $\Box$ are called the Laplacian and the d'Alembertian operator, respectively.

A.2 γ matrices

The Dirac 4×4 γ-matrices satisfy the following anticommutation relations,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I \tag{A.8}$$

where I is the 4×4 unit matrix. (A.8) leads to the following relations:

$$\begin{aligned} \gamma^\mu \gamma_\mu &= 4\,, & \text{(A.9)}\\ \gamma^\mu \gamma_\nu \gamma_\mu &= -2\gamma_\nu\,, & \text{(A.10)}\\ \gamma^\mu \gamma_\nu \gamma_\rho \gamma_\mu &= 4g_{\nu\rho}\,, & \text{(A.11)}\\ \gamma^\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\mu &= -2\gamma_\sigma \gamma_\rho \gamma_\nu\,. & \text{(A.12)} \end{aligned}$$

Define the chirality operator γ_5 as

$$\gamma^5 = \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3\,, \tag{A.13}$$

which satisfies

$$\gamma_5^2 = 1\,, \tag{A.14}$$

and

$$\{\gamma_5, \gamma^\mu\} = 0\,. \tag{A.15}$$

By using the Levi–Cività tensor which is totally antisymmetric, one can also write γ_5 as

$$\gamma_5 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\,, \tag{A.16}$$

where

$$\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{for even permutations of } 0,1,2,3, \\ -1 & \text{for odd permutations of } 0,1,2,3, \\ 0 & \text{otherwise.} \end{cases} \tag{A.17}$$

The Levi–Cività tensor satisfies the useful relations:

$$\begin{aligned} \varepsilon^{\alpha\lambda\mu\nu}\varepsilon_{\alpha\rho\sigma\tau} &= -\delta^\lambda_\rho(\delta^\mu_\sigma\delta^\nu_\tau - \delta^\mu_\tau\delta^\nu_\sigma) + \delta^\lambda_\sigma(\delta^\mu_\rho\delta^\nu_\tau - \delta^\mu_\tau\delta^\nu_\rho) \\ &\quad -\delta^\lambda_\tau(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho)\,, & \text{(A.18)}\\ \varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\rho\sigma} &= -2(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho)\,, & \text{(A.19)} \end{aligned}$$

$$\varepsilon^{\alpha\beta\gamma\mu}\varepsilon_{\alpha\beta\gamma\rho} = -6\delta^{\mu}_{\rho} \tag{A.20}$$

$$\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta} = -24 \, , \tag{A.21}$$

where δ^{μ}_{ν} is the Kronecker's delta,

$$\delta^{\mu}_{\nu} = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \, . \end{cases} \tag{A.22}$$

The scalar product of a γ-matrix and a 4-vector is given by

$$\gamma_\mu a^\mu \equiv \not{a} = \gamma^0 a^0 - \vec{\gamma} \cdot \vec{a} \tag{A.23}$$

and similarly

$$\gamma_\mu p^\mu = i\gamma_\mu \partial^\mu = i\not{\partial} = i\gamma^0 \partial_t + i\vec{\gamma} \cdot \nabla \, . \tag{A.24}$$

There are two kinds of useful representation of γ-matrices.
(1) γ^0-diagonal representation:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{A.25}$$

(2) γ_5-diagonal representation:

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{A.26}$$

where I is the 2×2 unit matrix and $\vec{\sigma}$ is the 2×2 Pauli matrix

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.27}$$

The spin tensor is defined as

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \, , \tag{A.28}$$

which, in the γ^0-digonal representation, leads to

$$\sigma^{ij} = \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \quad \text{and} \quad \sigma^{0i} = i \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{A.29}$$
$$(i, j, k = 1, 2, 3)$$

where ε^{ijk} is the totally antisymmetric tensor with $\varepsilon^{123} = 1$.

A.3 Trace theorems

In calculating the transition matrix elements, the following trace theorems are useful:

$$\mathrm{Tr}I = 4 \, , \tag{A.30}$$

$$\mathrm{Tr}\gamma_5 = 0 \, , \tag{A.31}$$

$$\mathrm{Tr}(\text{odd number of } \gamma' s) = 0 \, , \tag{A.32}$$

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu) = 4 g_{\mu\nu} \, , \tag{A.33}$$

$$\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) \, , \tag{A.34}$$

$$\mathrm{Tr}(\gamma_5 \gamma_\mu) = 0 \, , \tag{A.35}$$

$$\mathrm{Tr}(\gamma_5 \gamma_\mu \gamma_\nu) = 0 \, , \tag{A.36}$$

$$\mathrm{Tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho) = 0 \, , \tag{A.37}$$

$$\mathrm{Tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 4i\varepsilon_{\mu\nu\rho\sigma} \, , \tag{A.38}$$

The following relations are also useful:

$$\mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2) = 4\left[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - g^{\mu\nu}(p_1 \cdot p_2)\right] \tag{A.39}$$

$$\mathrm{Tr}\left[\gamma^\mu(1-\gamma_5)\not{p}_1 \gamma^\nu (1-\gamma_5)\not{p}_2\right] = 2\mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2) - 8i\varepsilon^{\mu\nu\alpha\beta} p_{1\alpha} p_{2\beta} \tag{A.40}$$

$$\mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_2)\mathrm{Tr}(\gamma_\mu \not{p}_3 \gamma_\nu \not{p}_4) = 32\left[(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)\right] \tag{A.41}$$

$$\mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu \gamma_5 \not{p}_2)\mathrm{Tr}(\gamma_\mu \not{p}_3 \gamma_\nu \gamma_5 \not{p}_4) = 32\left[(p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_4)(p_2 \cdot p_3)\right] \tag{A.42}$$

$$\mathrm{Tr}\left[\gamma^\mu(1-\gamma_5)\not{p}_1 \gamma^\nu(1-\gamma_5)\not{p}_2\right]\mathrm{Tr}\left[\gamma_\mu(1-\gamma_5)\not{p}_3 \gamma_\nu (1-\gamma_5)\not{p}_4\right\} = 256(p_1 \cdot p_3)(p_2 \cdot p_4), \tag{A.43}$$

A.4 Right- and left-handed Dirac spinors

The Dirac equation for a free fermion with mass m is given by

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \, , \tag{A.44}$$

where $\psi(x)$ represents the 4 component spinor. It is convenient to define chiral spinors (left- and right-handed spinors) as

$$\begin{aligned} \psi_L &= \frac{1-\gamma_5}{2}\psi = P_L \psi \qquad \left(P_L \equiv \frac{1-\gamma_5}{2}\right) , \\ \psi_R &= \frac{1-\gamma_5}{2}\psi = P_R \psi \qquad \left(P_R \equiv \frac{1-\gamma_5}{2}\right) , \end{aligned} \tag{A.45}$$

where P_L and P_R are projection operators of left-handed and right-handed spinors, respectively, and satisfy

$$\begin{aligned} &P_L + P_R = 1, \quad P_L^\dagger = P_L \ , \quad P_R^\dagger = P_R \ , \\ &P_L^2 = P_L, \quad P_R^2 = P_R \ , \quad P_L P_R = P_R P_L = 0 \ . \end{aligned} \tag{A.46}$$

Note that

$$\gamma_5 \psi_R = \psi_R \quad \text{and} \quad \gamma_5 \psi_L = -\psi_L \ . \tag{A.47}$$

Thus, ψ_L and ψ_R are eigenvectors of the chirality operator γ_5. Define the Dirac adjoint spinor $\bar{\psi} = \psi^\dagger \gamma^0$. Then we have the following important identities:

$$\bar{\psi} P_R = \bar{\psi}_L \ , \quad \bar{\psi} P_L = \bar{\psi}_R \ , \tag{A.48}$$

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R \ , \tag{A.49}$$

$$\bar{\psi}\gamma^\mu\gamma_5\psi = \bar{\psi}_R\gamma^\mu\psi_R - \bar{\psi}_L\gamma^\mu\psi_L \ , \tag{A.50}$$

$$\bar{\psi}\psi = \bar{\psi}_R\psi_L + \bar{\psi}_L\bar{\psi}_R \ . \tag{A.51}$$

A.5 Dirac equation in the γ_5-diagonal representation:

When we write the 4-component Dirac spinor in terms of 2-component Weyl spinors χ and η as

$$\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}, \tag{A.52}$$

we have the relations in the γ_5-diagonal representation

$$P_R\psi = \psi_R = \begin{pmatrix} \chi \\ 0 \end{pmatrix}, \quad P_L\psi = \psi_L = \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \tag{A.53}$$

where χ and η are unrelated 2-component spinors. In the γ_5-diagonal representation, the Dirac equation is written as

$$\begin{pmatrix} 0 & -i\frac{\partial}{\partial t} + i\vec{\sigma}\cdot\nabla \\ -i\frac{\partial}{\partial t} - i\vec{\sigma}\cdot\nabla & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix} - m \begin{pmatrix} \chi \\ \eta \end{pmatrix} = 0 \ , \tag{A.54}$$

which results in the following simultaneous equation

$$(-i\frac{\partial}{\partial t} - i\vec{\sigma}\cdot\nabla)\chi - m\eta \quad = \quad 0 \ , \tag{A.55a}$$

$$(-i\frac{\partial}{\partial t} + i\vec{\sigma}\cdot\nabla)\eta - m\chi \quad = \quad 0 \ . \tag{A.55b}$$

If $m = 0$, we have two independent Weyl equations,

$$i\frac{\partial}{\partial t}\chi = -i\vec{\sigma}\cdot\nabla\chi\,, \tag{A.56a}$$
$$i\frac{\partial}{\partial t}\eta = i\vec{\sigma}\cdot\nabla\eta\,. \tag{A.56b}$$

Assume the plane-wave,

$$\chi(x) = \tilde{\chi}(p)e^{-ipx}\,, \tag{A.57a}$$
$$\eta(x) = \tilde{\eta}(p)e^{-ipx}\,. \tag{A.57b}$$

Then we have

$$E\chi = +\vec{\sigma}\cdot\vec{p}\chi\,. \tag{A.58a}$$
$$E\eta = -\vec{\sigma}\cdot\vec{p}\eta\,, \tag{A.58b}$$

For positive energy solutions with $E = |\vec{p}| > 0$, we have

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\chi = +\chi\,, \tag{A.59a}$$
$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\eta = -\eta\,. \tag{A.59b}$$

Defining the helicity operator by

$$h = \frac{\vec{s}\cdot\vec{p}}{|\vec{p}|} = \frac{1}{2}\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}. \tag{A.60}$$

Then we can write

$$h\chi = +\frac{1}{2}\chi\,, \tag{A.61a}$$
$$h\eta = -\frac{1}{2}\eta\,, \tag{A.61b}$$

that is, helicity of χ and η is $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively.

If we use the 4-component notation for spin as $\vec{s} = \begin{pmatrix} \vec{\sigma}/2 & 0 \\ 0 & \vec{\sigma}/2 \end{pmatrix}$, we have

$$\begin{aligned} h\psi &= \vec{s}\cdot\hat{p}\psi = \begin{pmatrix} \vec{\sigma}\cdot\hat{p}/2 & 0 \\ 0 & \vec{\sigma}\cdot\hat{p}/2 \end{pmatrix}\begin{pmatrix} \chi \\ \eta \end{pmatrix} \\ &= \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \chi \\ \eta \end{pmatrix} = \frac{1}{2}\gamma_5\psi\,, \end{aligned} \tag{A.62}$$

where $\hat{p} = \frac{\vec{p}}{|\vec{p}|}$. Therefore, helicity h is the same as chirality γ_5 for massless fermions with positive energy. For negative-energy fermions(which correspond to antiparticles), helicity h has the opposite sign to chirality γ_5. If a fermion is massive ($m \neq 0$),

χ and η are still eigenstates of chirality, i.e.

$$\gamma_5 \begin{pmatrix} \chi \\ 0 \end{pmatrix} = \begin{pmatrix} \chi \\ 0 \end{pmatrix} \quad \text{and} \quad \gamma_5 \begin{pmatrix} 0 \\ \eta \end{pmatrix} = - \begin{pmatrix} 0 \\ \eta \end{pmatrix} \tag{A.63}$$

but they are no longer eigenstates of helicity h because of the non-zero mass term in the Dirac equation.

$$\vec{\sigma} \cdot \vec{p}\tilde{\chi} = +E\tilde{\chi} + m\tilde{\eta} \ . \tag{A.64a}$$

$$\vec{\sigma} \cdot \vec{p}\tilde{\eta} = -E\tilde{\eta} - m\tilde{\chi} \ , \tag{A.64b}$$

But for high energy particles with $E \gg m$, we can approximate $\frac{\vec{\sigma} \cdot \vec{p}}{E} \approx \frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}$, and then we have

$$h\chi = +\frac{1}{2}\chi + \frac{m}{2E}\eta \ , \tag{A.65a}$$

$$h\eta = -\frac{1}{2}\eta - \frac{m}{2E}\chi \ . \tag{A.65b}$$

By using the eigenstates of h, χ' and η', as defined as $h\chi' = +\frac{1}{2}\chi'$ and $h\eta' = -\frac{1}{2}\eta'$, respectively, one can write them as

$$\chi \approx \chi' + \frac{m}{E}\eta' \ , \tag{A.66a}$$

$$\eta \approx \eta' - \frac{m}{E}\chi' \ . \tag{A.66b}$$

That is, χ and η are given by the mixture of both helicity states where contribution of wrong sign helicity is order of m/E.

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Appendix B

Cross sections and Feynman rule

B.1 Cross sections

The scattering cross section or the transition probability for particle reactions is described by the S matrix as

$$S = 1 + iT \tag{B.1}$$

where T, being called the transition matrix, denotes the transition between initial i and final f states.

First, consider a 2-body collision, $a(p_a) + b(p_b) \to c(p_c) + d(p_d)$, as an example, where the momentum of each particle is given in parentheses. The transition matrix element for this process can be written as

$$T_{fi} = (2\pi)^4 \delta^4(p_a + p_b - p_c - p_d) N_a N_b N_c N_d M_{fi}. \tag{B.2}$$

For the time being, we consider the case of spinless particles. Extention to more general case including particles with spins is straightforward as described later. In (B.2), all dynamics for this transition are included in M_{fi} which is Lorentz invariant and called the invariant amplitude. N is a normalization factor which depends on the convention of the wave function normalization. Here we take the convention for normalizing $2E$ particles in a volume V, i.e. $N = \frac{1}{\sqrt{V}}$ by defining the free field wave function as

$$\phi(x) = \frac{1}{\sqrt{V}} e^{-ipx}, \tag{B.3}$$

which leads to $\int \rho dV = 2E$ with $\rho = i\left(\phi^* \partial_0 \phi - \partial_0 \phi^* \phi\right) = \frac{2E}{V}$. Substituting $N = \frac{1}{\sqrt{V}}$ for all particles involved into (B.2), we can write

$$T_{fi} = (2\pi)^4 \delta^4(p_a + p_b - p_c - p_d) \frac{1}{V^2} M_{fi}, \tag{B.4}$$

The transition probability per unit time and unit volume for this collision is given by

$$W_{fi} = \frac{|T_{fi}|^2}{TV}, \tag{B.5}$$

where T is the time interval of the interaction. By using the relation

$$\left[\delta^4(Q)\right]^2 \to \delta^4(Q)\frac{1}{(2\pi)^4}\int d^4x e^{iQx}(Q=0) = \delta^4(Q)\frac{VT}{(2\pi)^4}, \tag{B.6}$$

W_{fi} can be written as

$$W_{fi} = \frac{(2\pi)^4}{V^4}\delta^4(p_a + p_b - p_c - p_d)|M_{fi}|^2. \tag{B.7}$$

From (B.7), we can obtain the cross section σ by multipling it by the number of available final states and dividing it by the incident flux F. As is well known, the number of states in which a particle in a volume V has its momentum between $\vec{p} = (p_x, p_y, p_z)$ and $\vec{p} + d\vec{p} = (p_x + dp_x, p_y + dp_y, p_z + dp_z)$ is $\frac{Vd^3p}{(2\pi)^3}$. Since we are normalizing $2E$ particles in V, the number of states per one particle is $\frac{Vd^3p}{(2\pi)^3 2E}$. Then, the number of available final states for the process $a + b \to c + d$ is $\frac{Vd^3p_c}{(2\pi)^3 2E_c} \cdot \frac{Vd^3p_d}{(2\pi)^3 2E_d}$. The incident flux F can be calculated as follows; if we consider the process in the laboratory frame, the number of the incident particle a passing through unit area per unit time in this collision is $\frac{2E_a}{V}|\vec{v}_{in}|$, where $\vec{v}_{in}$ is the velocity of the incident particle a, while the number of the target particle b per unit volume is $\frac{2E_b}{V}$. Thus, the incident flux for the collision of a and b is $F = |\vec{v}_{in}|\frac{2E_a}{V}\frac{2E_b}{V}$. Then we can obtain the cross section for this process as follows;

$$\begin{aligned} d\sigma(a+b \to c+d) &= \frac{1}{|\vec{v}_{in}|4E_aE_b}\frac{1}{(2\pi)^2}\delta^4(p_a + p_b - p_c - p_d) \\ &\times \frac{d^3p_c}{2E_c}\frac{d^3p_d}{2E_d}|M(a+b \to c+d)|^2. \end{aligned} \tag{B.8}$$

The normalization volume V disappeared from (B.8). This is natural because V is not a physical parameter. Therefore, hereafter we set $V = 1$ which corresponds to take the wave function normalization as $N = 1$. This is the reason why we can assign 1 to the external line of the spinless particle in Feynman diagrams as shown below.

In the general collinear collision of a and b, $\vec{v}_{in}$(incoming velocity of a) in the incident flux $F = |\vec{v}_{in}|4E_aE_b$ in the laboratory frame is replaced by $\vec{v}_{rel}$(relative velocity of a and b). F can be rewritten into the Lorentz invariant form as

$$\begin{aligned} F &= |\vec{v}_{rel}|4E_aE_b = 4|\vec{v}_a - \vec{v}_b|E_aE_b \\ &= 4(|\vec{p}_a|E_a + |\vec{p}_b|E_b) = 4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2},. \end{aligned} \tag{B.9}$$

Now, the extension of this formula to the n particle production processes, $a+b \to 1+2+\cdots+n$, is straightforward,

$$\begin{aligned} d\sigma(a+b &\to 1+2+\cdots+n) \\ &= \frac{1}{4\sqrt{(p_a\cdot p_b)^2 - m_a^2 m_b^2}} \frac{1}{(2\pi)^{(3n-4)}} \delta^4(p_a+p_b-p_1-p_2-\cdots-p_n) \\ &\times \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\cdots\frac{d^3p_n}{2E_n}|M(a+b\to 1+2\cdots+n)|^2. \end{aligned} \tag{B.10}$$

One can easily generalize these formulas to processes including particles with spin by replacing $|M_{fi}|^2$ in (B.8) and (B.10) by $\sum_{spin}\overline{|M_{fi}|^2}$, which means that the spin degree of freedom must be summed for the final states and averaged for the initial states if the spin of initial particles is not polarized. In addition, for spin $\frac{1}{2}$ fermions the normalization convention for the wave function,

$$\bar{u}^{(s)}(p)u^{(s)}(p) = 2m, \quad \bar{v}^{(s)}(p)v^{(s)}(p) = -2m, \tag{B.11}$$

and also the energy projection operator,

$$\begin{aligned} \Lambda_+(p) &= \sum_s u^{(s)}(p)\bar{u}^{(s)}(p) = \not{p} + m, \\ \Lambda_-(p) &= -\sum_s v^{(s)}(p)\bar{v}^{(s)}(p) = -\not{p} + m, \end{aligned} \tag{B.12}$$

must be used in these cross section formulas.

When we write the cross section formula (B.10) as

$$d\sigma(a+b\to 1+2+\cdots+n) = \frac{\overline{|M_{fi}|^2}}{F} dR^{(n)}, \tag{B.13}$$

$dR^{(n)}$ is called the Lorentz invariant phase space which is given by

$$\begin{aligned} dR^{(n)} &= (2\pi)^4\delta^4(p_a+p_b-p_1-p_2-\cdots-p_n) \\ &\times \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\cdots\frac{d^3p_n}{(2\pi)^3 2E_n}. \end{aligned} \tag{B.14}$$

B.2 Decay width

The decay width of a particle can be calculated similarly; the decay width formula for a particle with mass M decaying into n particles in its rest frame is given by

$$\begin{aligned} d\Gamma(M &\to 1+2+\cdots+n) \\ &= \frac{1}{2M}\frac{1}{(2\pi)^{(3n-4)}}\overline{|M_{fi}|^2}\delta^4(p_a+p_b-p_1-p_2-\cdots-p_n) \\ &\times \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\cdots\frac{d^3p_n}{2E_n}. \end{aligned} \tag{B.15}$$

B.3 Feynman rule

The invariant amplitude M_{fi} for an specific process is given by the sum of Feynman diagrams corresponding to its process. They are usually composed of three kinds of factors, i.e. external lines, propagators(internal lines) and vertices, which are described below.

Feynman rule for QED

(1)External lines
- spin=0 bosons: For both incoming and outgoing bosons, assign 1.

- spin=$\frac{1}{2}$ fermions: For an incoming fermion or anti-fermion, assign $u^{(s)}(p)$ or $v^{(s)}(p)$, respectively. For an outgoing fermion or anti-fermion, assign $\bar{u}^{(s')}(p')$ or $\bar{v}^{(s')}(p')$, respectively.

- spin=1 photon: For an incoming photon, assign $\epsilon_\mu^{(\lambda)}(k)$. For an outgoing photon, assign $\epsilon_\mu^{*(\lambda')}(k')$

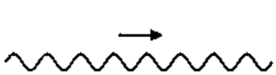
(2)Propagators
- Boson with spin=0:

$$= \frac{i}{p^2 - m^2}$$

- Fermion with spin=$\frac{1}{2}$:

$$= \frac{i}{\not{p} - m} = i\frac{\not{p} + m}{p^2 - m^2}$$

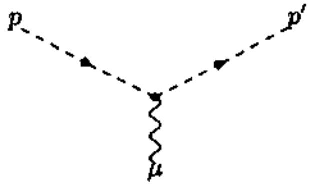
- Photon with spin=1:



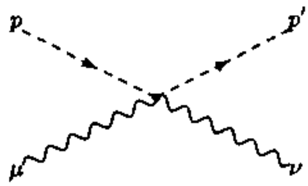
$$= \frac{i}{k^2}\left(-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)$$

where ξ is the gauge parameter. In the Feynman gauge, $\xi = 1$.

(3)Vertices
- spin=0 boson–photon vertex:

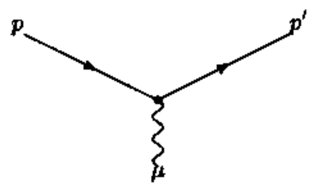


$-ie(p+p')_\mu$
(for charge $+e$)



$2ie^2 g_{\mu\nu}$

• spin=$\frac{1}{2}$ fermion–photon vertex:



$-ie\gamma_\mu$
(for charge $+e$)

Feynman rule for QCD

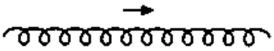
(1)External lines
• spin=$\frac{1}{2}$ quarks: For an incoming quark or anti-quark, assign $u^{(s)}(p)$ or $v^{(s)}(p)$, respectively. For an outgoing quark or anti-quark, assign $\bar{u}^{(s')}(p')$ or $\bar{v}^{(s')}(p')$, respectively. Though quarks have the flavor and color degree of freedom, it is not written explicitly in these spinors.

• spin=1 gluon: For an incoming gluon, assign $\epsilon_\mu^{(\lambda)}(k)\frac{\lambda^i}{2}$. For an outgoing gluon, assign $\epsilon_\mu^{*(\lambda')}(k')\frac{\lambda^i}{2}$. Here $\frac{\lambda^i}{2}$ $(i=1,2,\cdots,8)$ represent the color degree of freedom.

(2)Propagators
• Quark with spin=$\frac{1}{2}$:

$$= \frac{i}{\not{p}-m} = i\frac{\not{p}+m}{p^2-m^2}$$

• Gluon with spin=1:

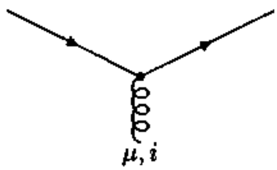


$$= \frac{i}{q^2}\left(-g^{\mu\nu} + (1-\xi)\frac{q^\mu q^\nu}{q^2}\right)\delta_{ij}$$

where ξ is the gauge parameter. In the Feynman gauge, $\xi = 1$.

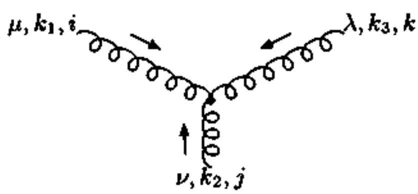
(3)Vertices

• Quark–gluon vertex:

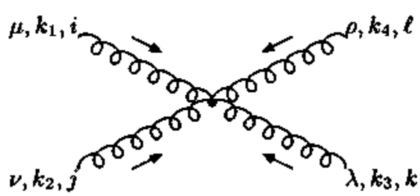


$$-ig_s\frac{\lambda^i}{2}\gamma_\mu$$

• Gluon–gluon vertex:



$$-g_s f_{ijk}[g_{\mu\nu}(k_1-k_2)_\lambda + g_{\nu\lambda}(k_2-k_3)_\mu + g_{\lambda\mu}(k_3-k_1)_\nu]$$



$$\begin{aligned}-ig_s^2[&f_{ijm}f_{k\ell m}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda})\\ &+f_{i\ell m}f_{jkm}(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho})\\ &+f_{ikm}f_{\ell jm}(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\lambda\rho})]\end{aligned}$$

Feynman rule for the GWS model of electroweak interactions

(1)External lines

• spin=0 Higgs bosons: For both incoming and outgoing Higgs bosons, assign 1.

• spin=$\frac{1}{2}$ leptons and quarks: For an incoming fermion or anti-fermion, assign $u^{(s)}(p)$ or $v^{(s)}(p)$, respectively. For an outgoing fermion or anti-fermion, assign $\bar{u}^{(s')}(p')$ or $\bar{v}^{(s')}(p')$, respectively. Electroweak interactions are not affected by color charges.

• spin=1 gauge bosons: For an incoming gauge boson, assign $\epsilon_\mu^{(\lambda)}(k)$. For an outgoing gauge boson, assign $\epsilon_\mu^{*(\lambda')}(k')$

(2)Propagators

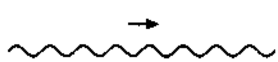
• Higgs boson with spin=0:

$$= \frac{i}{p^2 - \mu^2}$$

• Lepton and quark with spin=$\frac{1}{2}$:

$$= \frac{i}{\not{p} - m} = i\frac{\not{p} + m}{p^2 - m^2}$$

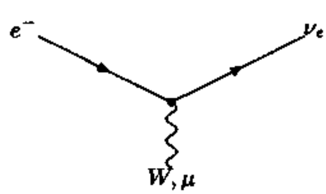
• Gauge boson with spin=1:



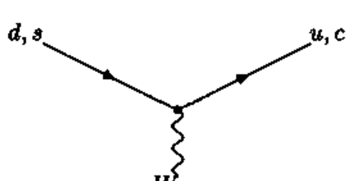
$$= \frac{i}{k^2 - M_V^2}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{M_V^2}\right)$$

(3)Vertices

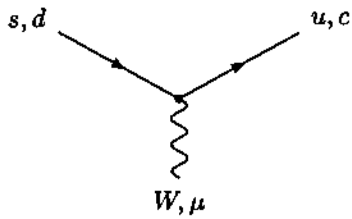
• Charged current vertex:



$$-i\frac{g}{\sqrt{2}}\gamma_\mu\frac{1-\gamma_5}{2}$$



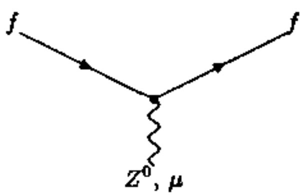
$$-i\frac{g}{\sqrt{2}}\cos\theta_C\gamma_\mu\frac{1-\gamma_5}{2}$$



$$\mp i \frac{g}{\sqrt{2}} \sin\theta_c \gamma_\mu \frac{1-\gamma_5}{2}$$

$$\begin{pmatrix} - & \text{is for } s\text{-}u\text{-}W \text{ vertex} \\ + & \text{is for } d\text{-}c\text{-}W \text{ vertex} \end{pmatrix}$$

• Neutral current vertex:



$$\frac{-ie}{\sin\theta_W \cos\theta_W} \gamma_\mu \left(a_L^f \frac{1-\gamma_5}{2} + a_R^f \frac{1+\gamma_5}{2} \right)$$

where

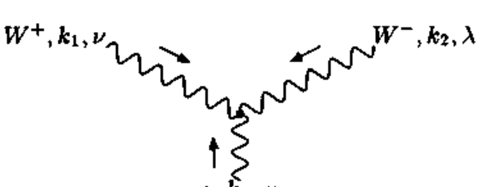
$$a_L^f = -\frac{1}{2} + \sin^2\theta_W, \quad a_R^f = \sin^2\theta_W, \quad \text{for } f = e^-, \mu^-, \tau^-$$

$$a_L^f = \frac{1}{2} - \frac{2}{3}\sin^2\theta_W, \quad a_R^f = -\frac{2}{3}\sin^2\theta_W, \quad \text{for } f = u, c, t$$

$$a_L^f = -\frac{1}{2} + \frac{1}{3}\sin^2\theta_W, \quad a_R^f = \frac{1}{3}\sin^2\theta_W, \quad \text{for } f = d, s, b.$$

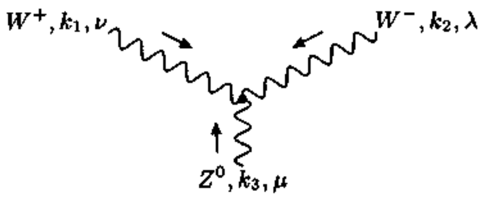
and for massless neutrinos, $a_L^f = \frac{1}{2}$, and $a_R^f = 0$.

• Gauge boson vertex:
1. Three point interaction vertex:
$\gamma W^+ W^-$ vertex:



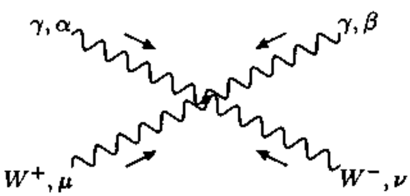
$$ie[g_{\nu\lambda}(k_1 - k_2)_\mu + g_{\lambda\mu}(k_2 - k_\gamma)_\nu + g_{\mu\nu}(k_\gamma - k_1)_\lambda]$$

$Z^0 W^+ W^-$ vertex:

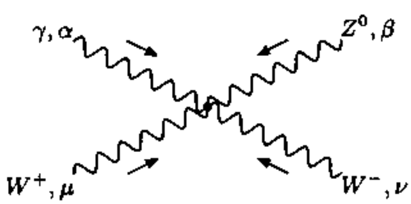


$$ie \cot\theta_W [g_{\nu\lambda}(k_1 - k_2)_\mu + g_{\lambda\mu}(k_2 - k_3)_\nu + g_{\mu\nu}(k_3 - k_1)_\lambda]$$

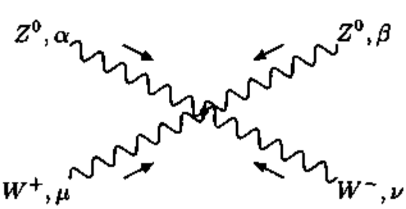
2. Four point interaction vertex:



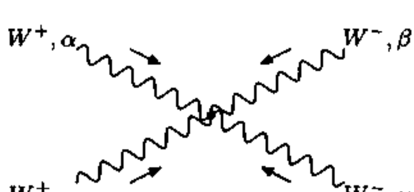
$$-ie^2(2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})$$



$$-ie^2 \cot\theta_W (2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})$$



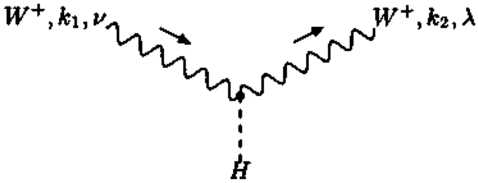
$$-ie^2 \cot^2\theta_W (2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})$$



$$\frac{ie^2}{\sin^2\theta_W}(2g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\alpha\nu} - g_{\mu\nu}g_{\alpha\beta})$$

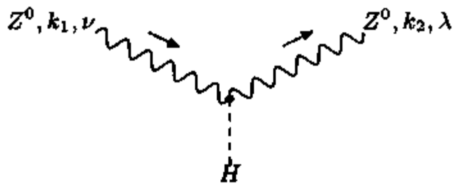
• Higgs boson (H) vertex:

1. HW^+W^- vertex:



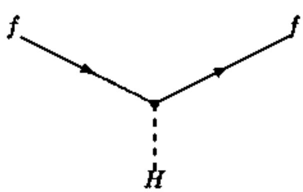
$$\frac{ie}{\sin\theta_W} M_W g_{\nu\lambda}$$

2. HZ^0Z^0 vertex:



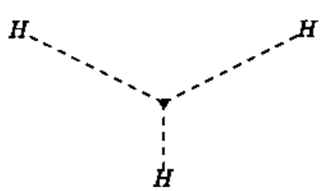
$$\frac{2ie}{\sin 2\theta_W} M_Z g_{\nu\lambda}$$

3. Fermion–Higgs Yukawa coupling vertex:



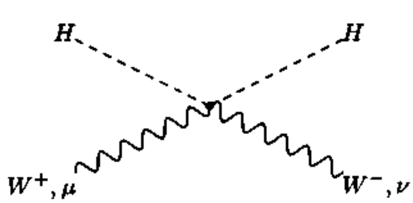
$$-\frac{ie}{2\sin\theta_W}\frac{m_f}{M_W}$$

4. Higgs boson self–coupling vertex:



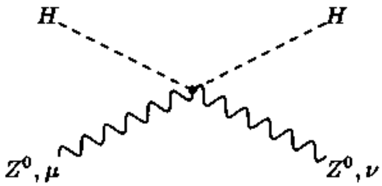
$$-i\frac{3\mu^2 e}{2M_W\cos\theta_W}$$

5. Four point interaction vertex including Higgs boson:
HHW^+W^- coupling:



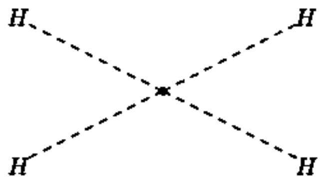
$$\frac{ie^2}{2\sin^2\theta_W} g_{\mu\nu}$$

HHZ^0Z^0 coupling:



$$\frac{2ie^2}{\sin^2 2\theta_W} g_{\mu\nu}$$

$HHHH$ coupling:



$$-\frac{3i\mu^2 e^2}{4M_W^2 \sin^2 \theta_W}$$

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Appendix C

Basics of the group theory

The Lagrangian is the fundamental object describing the dynamics of the physical system and is to be constructed so that the interactions are invariant under certain symmetry transformations. The requirement of invariance under symmetry transformations is a guiding principle for the construction of interacting field theories. Here we give a little mathematical basics of the symmetry, i.e. the group theory, in particular, an $SU(n)$ group and its algebra.

C.1 Group and representation

A group G is defined as a set of elements $\{a, b, c, \cdots\}$ with the follwing conditions,

(1) If a and b are in G, $a \cdot b$ is also in G.
(2) There is an associativity law, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, for any elements in G.
(3) There exists a unit element e in G, which satisfies $a \cdot e = e \cdot a = a$ for any a in G.
(4) There exists an inverse element a^{-1} in G for any a in G, such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Among many examples of groups, the n dimensional orthogonal group ($O(n)$) and special unitary group ($SU(n)$) are interesting in particle physics. $O(n)$ is the one in which the length $r^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ in the n dimensional real space is invariant under any rotation in this real space. $U(n)$ is the one in which the length defined by $s^2 = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$ in the n dimensional complex space is invariant under any rotation in this complex space and in addition, when the determinant of the transformation matrix U is 1, i.e. $\det U = 1$, it is called the $SU(n)$ group.

In case of $a \cdot b = b \cdot a$ for any elements a, b in G, the group is called the Abelian or commutative group, while in case of $a \cdot b \neq b \cdot a$, it is called the non-Abelian or non-commutative group. One–dimensional translation, two–dimensional rotation ($O(2)$) or phase transformation ($U(1)$) etc. belong to the Abelian group, while three-dimensional rotation ($O(3)$) or special unitary transformation ($SU(n)$ with

$n \geq 2$) etc. belong to the non-Abelian group. Specially interesting group in particle physics is the Lie group in which the group element is the analytic function of the continuous parameter.

Let us assume that there is a matrix or operator $M(a)$ for any elements a in G. Then, if $M(a)$ satisfys the conditions, $M(a)M(b) = M(ab)$, $M(a^{-1}) = M^{-1}(a)$ and $M(1) = 1$, then $M(a)$ is called representation of the group G.

C.2 $SU(n)$ group and Lie algebra

We are interested in the $SU(n)$ group defined in the n dimensional complex space, which is expressed by an $n \times n$ unitary matrix with its determinant being equal to 1

$$U^\dagger = U^{-1}, \quad \det U = 1. \tag{C.1}$$

Then, how many parametrs are there in U? As is well known, an $n \times n$ complex matrix has $2n^2$ real parameters. But there are n^2 constraints among these parameters because of the unitarity relation $U^\dagger U = 1$ and moreover, there is one additional constraint due to the condition of $\det U = 1$. Therefore, the number of the real parameters of the matrix U is $2n^2 - (n^2+1) = n^2 - 1$. For example, the unitary matrix for $SU(2)$ and $SU(3)$ has 3 and 8 parameters, respectively. Using these parameters θ^i ($i = 1, 2, \cdots, n^2-1$), the $n \times n$ unitary matrix U for $SU(n)$ is written by

$$U(\theta) = e^{-i\theta^i L^i} = e^{-i\vec{\theta}\cdot\vec{L}} \tag{C.2}$$

where L^i are the $n \times n$ matrices called the generators for the $SU(n)$ group. Here the summation over i is implied. Since U is unitary, L^i are hermitian matrices. In addition, L^i is traceless due to $\det U = 1$. Explicit expressions of L^i, for example, for $SU(2)$ are given by the 2×2 Pauli spin matrices $\frac{\sigma^i}{2}$ ($i = 1, 2, 3$) or, equivalently, 2×2 isospin matrices $\frac{\tau^i}{2}$ ($i = 1, 2, 3$) with

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{C.3}$$

and for $SU(3)$, they are given by the 3×3 Gell-Mann matrices $\frac{\lambda^i}{2}$ ($i = 1, 2, \cdots, 8$) with

$$\begin{aligned}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\end{aligned}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{C.4}$$

As explicitly shown above, the $SU(2)$ group has one diagonal matrix, $\frac{\tau^3}{2}$, and the $SU(3)$ group has two diagonal matrices, $\frac{\lambda^3}{2}$ and $\frac{\lambda^8}{2}$. The number of diagonal matrices is equal to the rank of the group. The $SU(n)$ group has $n-1$ diagonal generators and is a group of rank $n-1$.

From the group property of U, we can see that these generators satisfy the following algebra which is called the Lie algebra

$$[L^i, L^j] = i f_{ijk} L^k, \quad (i, j, k = 1, 2, \cdots, n^2-1) \tag{C.5}$$

where f_{ijk}, being antisymmetric constants for exchange of any pairs of i, j and k, are called the structure constants of the group. Here the summation over k is implied. This statement can be proved as follows; let us start with the follwing relation for non-commuting operators A and B,

$$e^A \cdot e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots\cdots}. \tag{C.6}$$

Now let A and B to be given by $A = i\alpha^i L^i = i\vec{\alpha}\cdot\vec{L}$ and $B = i\beta^i L^i = i\vec{\beta}\cdot\vec{L}$ with n^2-1 parameters α^i and β^i, respectively, where the summation over i is implied. Then, since $U(\alpha) = e^A$ and $U(\beta) = e^B$ are the elements of $SU(n)$, $U(\alpha)\cdot U(\beta) = e^A \cdot e^B$ is also the element of $SU(n)$ because of the group property. Hence, $e^A \cdot e^B$ must be also written by e^C with $C = i\vec{\gamma}\cdot\vec{L}$. Therefore, the commutator $[L^i, L^j]$ must be linearly related to the sum of the generators as described in (C.5). The structure constants f_{ijk} are determined from the associativity law of the group.

The Jacobi identity

$$[L^i, [L^j, L^k]] + [L^j, [L^k, L^i]] + [L^k, [L^i, L^j]] = 0, \tag{C.7}$$

with the Lie algebra (C.5) leads to the relation

$$f_{jk\ell} f_{i\ell m} + f_{ki\ell} f_{j\ell m} + f_{ij\ell} f_{k\ell m} = 0. \tag{C.8}$$

Then, if we identify the matrices T^i as $(T^i)_{jk} = -i f_{ijk}$, (C.8) results in

$$[T^i, T^j] = i f_{ijk} T^k, \tag{C.9}$$

which is just the same algebra as (C.5). That is to say, the generators being just equal to the structure constants of the group satisfy the Lie algebra. In other words, the structure constants themselves generate a representation of the Lie algebra. The representation generated by the structure constants is called the "adjoint representation". The dimention of the adjoint representation is just the number of generators, which is n^2-1 for $SU(n)$.

C.3 Representation in $SU(n)$ group

Let us consider a field ϕ composed of n complex fields φ_a $(a = 1, 2, \cdots, n)$ and write it by a column vector,

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \cdot \\ \cdot \\ \cdot \\ \varphi_n \end{pmatrix}. \tag{C.10}$$

If ϕ changes into ϕ' under the $SU(n)$ transformation as

$$\phi \to \phi' = U\phi, \text{ or } \varphi_a \to \varphi'_a = U_a^b \varphi_b \equiv U_{ab}\varphi_b, \tag{C.11}$$

φ_a is said to be in the "fundamental representation" of $SU(n)$ and denoted by **n**. Now, let us consider an infinitesimal transformation

$$U(\theta) = e^{-i\vec{\theta}\cdot\vec{L}} \simeq 1 - i\vec{\theta}\cdot\vec{L}, \quad \text{for} \quad \theta \ll 1. \tag{C.12}$$

Then, we can write the field transformation as

$$\varphi_a \to \varphi'_a \simeq \varphi_a - \delta\varphi_a, \tag{C.13}$$

with

$$\delta\varphi_a = i(\vec{\theta}\cdot\vec{L})_a^b \varphi_b = i\epsilon_a^b \varphi_b. \tag{C.14}$$

Next, let us introduce the conjugate fundamental representation $\mathbf{n}^*$. If we take the complex conjugate of (C.11) and use the relation $(U^*)_a^b \equiv (U^*)_{ab} = (U^{*T})_{ba} = (U^\dagger)_{ba} \equiv (U^\dagger)_b^a$, where $U^\dagger$ is the hermite conjugate of U, then one can obtain the transformation of the congujate states φ_a^* as

$$\varphi_a^* \to \varphi_a'^* = (U^*)_a^b \varphi_b^* = \varphi_b^*(U^\dagger)_b^a. \tag{C.15}$$

Now, it is convenient to introduce the upper indices to these conjugate fields as $\varphi^a \equiv \varphi_a^*$. Then we can rewrite (C.15) as

$$\varphi^a \to \varphi'^a = \varphi^b(U^\dagger)_b^a, \tag{C.16}$$

The representation for those fields φ^a is called the "conjugate fundamental representation" and denoted by $\mathbf{n}^*$. Since the infinitesimal transformation associated with $U^*(\theta)$ is $U^*(\theta) \simeq 1 + i\theta^i L^{i*} = 1 - i\theta^i(-L^{i*})$, the conjugate representation $\varphi^a = \varphi_a^*$ are generated by $-L^{i*}$. Therefore, all eigenvectors for generators L^i of diagonal matrices are still eigenvectors of $-L^{i*}$ but of opposite sign. Regarding this property, the $SU(2)$ group is specially interesting because it has only one diagonal matrix $\frac{\tau^3}{2}$. Then for $SU(2)$, we can expect that the **2** and $\mathbf{2}^*$ representation is equivalent. In fact, this is the case as shown in the following. Let us look for a matrix A so that $A\phi$ transforms, like ϕ in (C.11), as $A\phi^* \to (A\phi^*)' = U(A\phi^*)$ or $\phi^* \to \phi'^* = (A^{-1}UA)\phi^* = U^*\phi^*$. Thus, we must have $A^{-1}UA = U^*$, that is, $A^{-1}\frac{\tau^i}{2}A = -\frac{\tau^{i*}}{2}$ $(i = 1,2,3)$ since $U \simeq 1 - i\theta^i\frac{\tau^i}{2}$ for $SU(2)$. This can be possible, when we choose $A = i\tau^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Actually, we can see that $A\phi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix} = \begin{pmatrix} \varphi^2 \\ -\varphi^1 \end{pmatrix}$ transforms like $\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$, i.e. $\mathbf{2}^* \simeq \mathbf{2}$.

Now, let us consider the direct product of $\mathbf{n}$ and $\mathbf{n}^*$ representations.

$$\varphi_a \otimes \chi^b = (\varphi_a\chi^b - \frac{1}{n}\delta_a^b\varphi_c\chi^c) + \frac{1}{n}\delta_a^b\varphi_c\chi^c. \tag{C.17}$$

By taking the trace of (C.17), we find $\mathrm{Tr}(\varphi_a \otimes \chi^b) = \varphi_a\chi^a$. Hence the 1st term of the right-hand side of (C.17) is the traceless matrix with $n^2 - 1$ components,

$$T_a^b = \varphi_a\chi^b - \frac{1}{n}\delta_a^b\varphi_c\chi^c, \tag{C.18}$$

and denoted by $\mathbf{n^2 - 1}$. The 2nd term is just the trace term having only one component,

$$S_a^b = \frac{1}{n}\delta_a^b\varphi_c\chi^c, \tag{C.19}$$

and is denoted by $\mathbf{1}$. From (C.11) and (C.16), we see that the transformation law of T and S under the $SU(n)$ transformation is given as follows;

$$T_a^b \to T_a'^b = U_a^m\varphi_m\chi^n U_n^{\dagger b} - \frac{1}{n}\delta_a^b U_c^m\varphi_m\chi^n U_n^{\dagger c}, \tag{C.20}$$

and

$$S_a^b \to S_a'^b = \frac{1}{n}\delta_a^b U_c^m\varphi_m\chi^n U_n^{\dagger c}. \tag{C.21}$$

Using the unitarity condition $U_c^m U_n^{\dagger c} = \delta_n^m$ where the summation over c is implied, these transformation law can be written in the matrix form as

$$T \to T' = UTU^\dagger, \tag{C.22}$$

and

$$S \to S' = S. \tag{C.23}$$

As described above, the matrix T_a^b is an $n\times n$ traceless matrix with n^2-1 components and thus it can be expanded by the n^2-1 traceless generators L^i $(i = 1,2,\cdots,n^2-1)$ of the $SU(n)$ group as

$$T_a^b = A^i(L^i)_a^b, \quad (i = 1,2,\cdots,n^2-1) \tag{C.24}$$

where A^i are the parameters of the $SU(n)$ group. Here the summation over i is implied. Now, let us consider the transformation law of A^i under the $SU(n)$ transformation. Under an infinitesimal transformation

$$\varphi_a \to \varphi'_a = \varphi_a - i\epsilon_a^b\varphi_b, \quad \varphi^a \to \varphi'^a = \varphi^a - i\varphi^b\epsilon_b^a, \tag{C.25}$$

the transformation law (C.22) leads to

$$T \to T' = T - i[\epsilon, T], \quad (\epsilon = \alpha^j L^j, \ \ T = A^k L^k) \tag{C.26}$$

which can be written as

$$\begin{aligned} A^i L^i \to A'^i L^i &= A^i L^i - i\alpha^j [L^j, L^k] A^k \\ &= A^i L^i - i\alpha^j (i f_{jki} L^i) A^k. \end{aligned} \tag{C.27}$$

That is,

$$A'^i = A^i + f_{ijk}\alpha^j A^k, \tag{C.28}$$

or by identifying $-if_{jik} = (T^j)_{ik}$, which is the adjoint representation, A^i transform as

$$\begin{aligned} \delta A^i &= f_{ijk}\alpha^j A^k = -i(-if_{jik})\alpha^j A^k \\ &= -i(T^j)_{ik}\alpha^j A^k = -i(\vec{T}\cdot\vec{\alpha})_{ik} A^k. \end{aligned} \tag{C.29}$$

In these equations the summation over the same subscripts is implied.

In summary, the T_a^b which transform according to (C.22) or (C.26) and are given by (C.24), have $n^2 - 1$ components and are called the "adjoint representation" of the $SU(n)$ group denoted by $\mathbf{n^2 - 1}$, while S_a^b is invariant under the $SU(n)$ transformation, that is, it transforms as a singlet and denoted by $\mathbf{1}$. We can decompose the product of $\mathbf{n}$ and $\mathbf{n^*}$ representation into $\mathbf{n^2 - 1}$ and $\mathbf{1}$,

$$\mathbf{n} \otimes \mathbf{n}^* = \mathbf{n^2 - 1} \oplus \mathbf{1}. \tag{C.30}$$

Appendix D

C, P and T transformation

In addition to the continuous transformations, there are several important discrete symmetries in physics. Here we concentrate our attention on space inversion (P), time inversion (T) and charge conjugation (C). In view of the field-theoretical model building, it is important to study whether or not the physical system is invariant under these transformations, and if not, how and in what degree these symmetries are violated.

D.1 Parity or space inversion P

A parity inversion is the reflection of a spatial plane. This is equivalent to a space refelection followed by a rotation with an angle π about an axis perpendicular to that plane after the space reflection. Thus a parity operation is equivalent to a space inversion. A parity operation is defined as

$$P: \quad \vec{x} \to \vec{x}' = -\vec{x}, \quad t \to t' = t. \tag{D.1}$$

Invariance of a physical system under parity transformation means that any interactions in the system should be left-right symmetric.

Here we just list up the transformation rule of fields under parity operation. For simplicity, we fix the phase factor to be $\eta_P = 1$.

$$\text{Scalar field} \quad : \quad \phi(\vec{x},t) \to \phi^P(\vec{x}',t') = \phi(-\vec{x},t), \tag{D.2}$$

$$\text{Pseudoscalar field} \quad : \quad \eta(\vec{x},t) \to \eta^P(\vec{x}',t') = -\eta(-\vec{x},t), \tag{D.3}$$

$$\text{Dirac field} \quad : \quad \begin{cases} \psi(\vec{x},t) \to \psi^P(\vec{x}',t') = \gamma_0\psi(-\vec{x},t), \\ \bar{\psi}(\vec{x},t) \to \bar{\psi}^P(\vec{x}',t') = \bar{\psi}(-\vec{x},t)\gamma_0, \end{cases} \tag{D.4}$$

$$\text{Vector field} \quad : \quad V^\mu(\vec{x},t) \to V^{\mu^P}(\vec{x}',t') = V_\mu(-\vec{x},t). \tag{D.5}$$

Furthermore, the bilinear covariants (S(scalar), P(pseudoscalar), V(vector), A(axial-vector), T(tensor)) composed of Dirac fields transform as

$$\mathrm{S} \quad : \quad \bar{\psi}_a(\vec{x},t)\psi_b(\vec{x},t) \to \bar{\psi}_a(-\vec{x},t)\psi_b(-\vec{x},t), \tag{D.6}$$

$$\mathrm{P} \;:\; \bar{\psi}_a(\vec{x},t)\gamma^5\psi_b(\vec{x},t) \to -\bar{\psi}_a(-\vec{x},t)\gamma^5\psi_b(-\vec{x},t), \tag{D.7}$$

$$\mathrm{V} \;:\; \bar{\psi}_a(\vec{x},t)\gamma^\mu\psi_b(\vec{x},t) \to \bar{\psi}_a(-\vec{x},t)\gamma_\mu\psi_b(-\vec{x},t), \tag{D.8}$$

$$\mathrm{A} \;:\; \bar{\psi}_a(\vec{x},t)\gamma^5\gamma^\mu\psi_b(\vec{x},t) \to -\bar{\psi}_a(-\vec{x},t)\gamma^5\gamma_\mu\psi_b(-\vec{x},t), \tag{D.9}$$

$$\mathrm{T} \;:\; \bar{\psi}_a(\vec{x},t)\sigma^{\mu\nu}\psi_b(\vec{x},t) \to \bar{\psi}_a(-\vec{x},t)\sigma_{\mu\nu}\psi_b(-\vec{x},t). \tag{D.10}$$

An important example of parity transformation is the one for the weak left-handed current defined by $J_\mu^{(W)} = \bar{\psi}_{aL}(\vec{x},t)\gamma_\mu\psi_{bL}(\vec{x},t)$, where the left- and right-handed fermion fields transform under parity inversion as

$$\begin{cases} \psi_{L,R}(\vec{x},t) \to (\psi^P)_{L,R}(\vec{x},t) = \gamma_0\psi_{R,L}(-\vec{x},t), \\ \bar{\psi}_{L,R}(\vec{x},t) \to (\bar{\psi}^P)_{L,R}(\vec{x},t) = \bar{\psi}_{R,L}(-\vec{x},t)\gamma_0. \end{cases} \tag{D.11}$$

Then, the weak left-handed current $J_\mu^{(W)}$ transforms as

$$\begin{aligned} J_\mu^{(W)} = \bar{\psi}_{aL}(\vec{x},t)\gamma_\mu\psi_{bL}(\vec{x},t) \;&\to\; \bar{\psi}_{aR}(-\vec{x},t)\gamma_0\gamma_\mu\gamma_0\psi_{bR}(-\vec{x},t) \\ &=\; \bar{\psi}_{aR}(-\vec{x},t)\gamma^\mu\psi_{bR}(-\vec{x},t), \end{aligned} \tag{D.12}$$

where we used the relation, $\gamma_0\gamma_\mu\gamma_0 = \gamma_\mu^\dagger = \gamma^\mu$.

D.2 Time inversion T

A time inversion is a reflection of time as defined as

$$T: \quad \vec{x} \to \vec{x}' = \vec{x}, \quad t \to t' = -t. \tag{D.13}$$

Here we just list up the transformation rule of fields under time inversion. For simplicity, we fix the phase to be $\eta_T = 1$.

$$\text{Scalar field} \;:\; \phi(\vec{x},t) \to \phi^T(\vec{x}',t') = \phi(\vec{x},-t), \tag{D.14}$$

$$\text{Pseudoscalar field} \;:\; \eta(\vec{x},t) \to \eta^T(\vec{x}',t') = -\eta(\vec{x},-t), \tag{D.15}$$

$$\text{Dirac field} \;:\; \begin{cases} \psi(\vec{x},t) \to \psi^T(\vec{x}',t') = \gamma^3\gamma^1\psi(\vec{x},-t), \\ \bar{\psi}(\vec{x},t) \to \bar{\psi}^T(\vec{x}',t') = \psi^T(\vec{x},-t)\gamma^1\gamma^3, \end{cases} \tag{D.16}$$

$$\text{Vector field} \;:\; V^\mu(\vec{x},t) \to V^{\mu^T}(\vec{x}',t') = V_\mu(\vec{x},-t). \tag{D.17}$$

One should not be confused at the superscript T to Dirac fields in (D.16); the superscript T on the left-hand side means the time-reversed field, while the one on the right-hand side does the transposed field.

Furthermore, the bilinear covariants composed of Dirac fields transform as

$$\mathrm{S} \;:\; \bar{\psi}_a(\vec{x},t)\psi_b(\vec{x},t) \to \bar{\psi}_a(\vec{x},-t)\psi_b(\vec{x},-t), \tag{D.18}$$

$$\mathrm{P} \;:\; \bar{\psi}_a(\vec{x},t)\gamma^5\psi_b(\vec{x},t) \to -\bar{\psi}_a(\vec{x},-t)\gamma^5\psi_b(\vec{x},-t), \tag{D.19}$$

$$\mathrm{V} \;:\; \bar{\psi}_a(\vec{x},t)\gamma^\mu\psi_b(\vec{x},t) \to \bar{\psi}_a(\vec{x},-t)\gamma_\mu\psi_b(\vec{x},-t), \tag{D.20}$$

$$\mathrm{A} \;:\; \bar{\psi}_a(\vec{x},t)\gamma^5\gamma^\mu\psi_b(\vec{x},t) \to \bar{\psi}_a(\vec{x},-t)\gamma^5\gamma_\mu\psi_b(\vec{x},-t), \tag{D.21}$$

$$\text{T} \quad : \quad \bar{\psi}_a(\vec{x},t)\sigma^{\mu\nu}\psi_b(\vec{x},t) \to -\bar{\psi}_a(\vec{x},-t)\sigma_{\mu\nu}\psi_b(\vec{x},-t). \tag{D.22}$$

D.3 Charge conjugation C

In addition to the space-time symmetries discussed above, there are other symmetries called internal symmetries. Among many such internal symmetries, we are here interested in the charge conjugation. This is a discrete transformation of particles into antiparticles, in which the electric charge and other quantum numbers of particles are reversed without changing any kinematic attributes.

Here we just list up the transformation rule of fields under charge conjugation. For simplicity, we fix the phase to be $\eta_C = 1$.

$$\text{Scalar field} \quad : \quad \phi(\vec{x},t) \to \phi^C(\vec{x}',t') = \phi^\dagger(\vec{x},t), \tag{D.23}$$

$$\text{Pseudoscalar field} \quad : \quad \eta(\vec{x},t) \to \eta^C(\vec{x}',t') = \eta^\dagger(\vec{x},t), \tag{D.24}$$

$$\text{Dirac field} \quad : \quad \begin{cases} \psi(\vec{x},t) \to \psi^C(\vec{x}',t') = C\bar{\psi}^T(\vec{x},t), \\ \bar{\psi}(\vec{x},t) \to \bar{\psi}^C(\vec{x}',t') = -\psi^T(\vec{x},t)C^{-1}, \end{cases} \tag{D.25}$$

$$\text{Vector field} \quad : \quad V^\mu(\vec{x},t) \to V^{\mu^C}(\vec{x}',t') = -V^{\dagger\mu}(\vec{x},t), \tag{D.26}$$

where the charge conjugation operator C in (D.25) is explicitly written by $C = i\gamma^2\gamma^0$ and satisfies the relations, $C^{-1}\gamma^\mu C = -\gamma^{\mu^T}$ and $C = -C^{-1} = -C^\dagger = -C^T$.

Furthermore, the bilinear covariants composed of Dirac fields transform as

$$\text{S} \quad : \quad \bar{\psi}_a(\vec{x},t)\psi_b(\vec{x},t) \to \bar{\psi}_b(\vec{x},t)\psi_a(\vec{x},t), \tag{D.27}$$

$$\text{P} \quad : \quad \bar{\psi}_a(\vec{x},t)\gamma^5\psi_b(\vec{x},t) \to \bar{\psi}_b(\vec{x},t)\gamma^5\psi_a(\vec{x},t), \tag{D.28}$$

$$\text{V} \quad : \quad \bar{\psi}_a(\vec{x},t)\gamma^\mu\psi_b(\vec{x},t) \to -\bar{\psi}_b(\vec{x},t)\gamma^\mu\psi_a(\vec{x},t), \tag{D.29}$$

$$\text{A} \quad : \quad \bar{\psi}_a(\vec{x},t)\gamma^5\gamma^\mu\psi_b(\vec{x},t) \to \bar{\psi}_b(\vec{x},t)\gamma^5\gamma^\mu\psi_a(\vec{x},t), \tag{D.30}$$

$$\text{T} \quad : \quad \bar{\psi}_a(\vec{x},t)\sigma^{\mu\nu}\psi_b(\vec{x},t) \to -\bar{\psi}_b(\vec{x},t)\sigma^{\mu\nu}\psi_a(\vec{x},t). \tag{D.31}$$

Now, a comment on the transformation of the left- and right-fields under charge conjugation is useful. Applying the transformation rule (D.25) to the left-handed Dirac spinor, we can obtain the following result,

$$\begin{aligned} (\psi_L)^C &= C\bar{\psi}_L^T = C\left[(\frac{1-\gamma_5}{2}\psi)^\dagger\gamma^0\right]^T \\ &= C\left[\psi^\dagger\frac{1-\gamma_5}{2}\gamma^0\right]^T = C\gamma^0\frac{1-\gamma_5}{2}\psi^* \\ &= \frac{1+\gamma_5}{2}C\gamma^0\psi^* = \frac{1+\gamma_5}{2}C\bar{\psi}^T \\ &= \frac{1+\gamma_5}{2}\psi^C = (\psi^C)_R \equiv \psi_R^C. \end{aligned} \tag{D.32}$$

Similarly we obtain the result, $(\psi_R)^C = (\psi^C)_L \equiv \psi_L^C$, for the right-handed spinor. Furthermore, we can also obtain the similar result even for these adjoint spinors.

We summarized them as

$$\begin{cases} \psi_{L,R} \to (\psi^C)_{L,R} = (\psi_{R,L})^C = C\bar{\psi}^T_{R,L}, \\ \bar{\psi}_{L,R} \to (\bar{\psi}^C)_{L,R} = (\bar{\psi}_{R,L})^C = \psi^T_{R,L}C. \end{cases} \tag{D.33}$$

D.4 CP transformation

We are here interested in the CP transformation properties of Dirac spinors and their bilinear covariants. A CP inversion is the direct product of parity operation and charge conjugation. Then, from the transformation rules described above, we obtain the following rules for Dirac spinors,

$$\begin{cases} \psi(\vec{x},t) \to \psi^{CP} = \gamma_0\psi^C(-\vec{x},t) = \gamma_0 C\bar{\psi}^T(-\vec{x},t), \\ \bar{\psi}(\vec{x},t) \to \bar{\psi}^{CP}(\vec{x},t) = \bar{\psi}^C(-\vec{x},t)\gamma_0 = \psi^T(-\vec{x},t)C\gamma_0, \end{cases} \tag{D.34}$$

and furthermore, combining (D.11) and (D.33), similar rules are derived for the left- and right-handed spinors as

$$\begin{cases} \psi_{L,R}(\vec{x},t) \to (\psi^{CP})_{L,R} = \gamma_0(\psi^C)_{R,L}(-\vec{x},t) = \gamma_0 C\bar{\psi}^T_{L,R}(-\vec{x},t), \\ \bar{\psi}_{L,R}(\vec{x},t) \to (\bar{\psi}^{CP})_{L,R}(\vec{x},t) = (\bar{\psi}^C)_{R,L}(-\vec{x},t)\gamma_0 = \psi^T_{L,R}(-\vec{x},t)C\gamma_0. \end{cases} \tag{D.35}$$

By using these transformation rules, we can obtain the CP transformation of the weak left-handed current as

$$\begin{aligned} J_{\mu(ab)} = \bar{\psi}_{aL}(\vec{x},t)\gamma_\mu\psi_{bL}(\vec{x},t) &\to \psi^T_{aL}C\gamma_0\gamma_\mu\gamma_0 C\bar{\psi}^T_{bL}(-\vec{x},t) \\ &= \psi^T_{aL}(-\vec{x},t)\gamma^{\mu^T}\bar{\psi}^T_{bL} \\ &= -\bar{\psi}_{bL}(-\vec{x},t)\gamma^\mu\psi_{aL}(-\vec{x},t) \\ &= -J^\mu_{(ba)}(-\vec{x},t), \end{aligned} \tag{D.36}$$

where the minus sign in the last two equations originates from the interchange of fermion fields in taking the transposition.

Let us consider the transformation of the charged current weak interaction under CP inversion. The Lagrangian is given in (4.164) as

$$\mathcal{L}^{(Q)}_{CC} = \frac{g}{\sqrt{2}}\bar{U}'_L\gamma^\mu V D'_L W^+_\mu + \frac{g}{\sqrt{2}}\bar{D}'_L V^\dagger\gamma^\mu U'_L W^-_\mu, \tag{D.37}$$

where V is the unitary matrix, being the Cabibbo–Kobayashi–Maskawa(CKM) matrix for the case of 3 generations, and $W^\pm_\mu$ is the charged weak boson fields defined by (4.55). If we write the charged currents as $J^{\mu-} \equiv \bar{U}'_L\gamma^\mu V D'_L$ and $J^{\mu+} \equiv \bar{D}'_L\gamma^\mu V^\dagger U'_L$, we obtain the CP transformed currents as

$$\begin{cases} (J^{\mu-})^{CP} = -\bar{D}'_L\gamma_\mu V^T U'_L, \\ (J^{\mu+})^{CP} = -\bar{U}'_L\gamma_\mu V^* D'_L. \end{cases} \tag{D.38}$$

Furthermore, the application of C and P inversions to the vecor field $W^{\pm}_{\mu}$ yields

$$W^{\pm}_{\mu} \to W^{CP\pm}_{\mu} = -W^{\mu\mp}. \tag{D.39}$$

Then, we get the following transformation,

$$\begin{aligned} &\int d^4x(\bar{U}'_L\gamma^{\mu}VD'_LW^{+}_{\mu} + \bar{D}'_L\gamma^{\mu}V^{\dagger}U'_LW^{-}_{\mu}) \\ \to &\int d^4x(\bar{D}'_L\gamma_{\mu}V^TU'_LW^{\mu-} + \bar{U}'_L\gamma_{\mu}V^*D'_LW^{\mu+}). \end{aligned} \tag{D.40}$$

Therefore, if $V = V^*$, i.e. V is real, the action is invariant under CP inversion, that is, CP is conserved. On the contrary, if $V \neq V^*$, i.e. V is not real, then CP is violated.

Appendix E
The quark model

In modern particle physics, the symmetry and the constituent of matter having its symmetry are powerful tools to understand the structure of matter and its physics. One such example is the proton and neutron which are the constituents of all nuclei and possess the $SU(2)$ symmetry of isospin. Various properties of nuclei are well understood by the dynamics of protons and neutrons having the $SU(2)$ symmetry of isospin. Now, it is known that there are more than 300 hadrons, including the proton and the neutron. Then, it is natural to expect that there must be new fundamental constituents at a deeper level in Nature which build those hadrons systematically based on new underlying symmetries. In 1964, Gell-Mann and Zweig (Gell-Mann, 1964; Zweig, 1964) proposed such a new constituent model of hadrons called the quark model, in which the hadrons are beautifully classified with the $SU(3)$ symmetry. The idea of quarks as the fundamental constituents of hadrons has been developed with a lot of theoretical and experimental efforts in 1960s and 1970s, and the quark model was established as an important component of the standard model of modern particle physics. Here we would like to discuss the quark model.

E.1 Isospin symmetry

Before discussing the quark model, let us begin by briefly reviewing the isospin symmetry. Nuclei are bound states of protons and neutrons and their nuclear force is issued from pion exchange. Interestingly the nuclear force does not distinguish the proton and the neutron. The proton mass $m_p = 938.27200 \pm 0.00004$ MeV is very close to the neutron mass $m_n = 939.56533 \pm 0.00004$ MeV and their small mass difference is considered to be due to the electromagnetic interaction; a proton has a positive charge, while a neutron is neutral. This situation naturally suggests that the proton and the neutron are two manifestations of the same particle, called the nucleon, and one can write it as a doublet $\mathbf{2} = \begin{pmatrix} p \\ n \end{pmatrix}$ of the fundamental representation of the $SU(2)$ group, that is, the proton and neutron are represented in the isospin space as

$$|p\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ (I^3 = +\frac{1}{2}); \quad |n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ (I^3 = -\frac{1}{2}) \tag{E.1}$$

which are just the same representation as the spin-up and spin-down states of spin $\frac{1}{2}$ particles, respectively. Similarly, the masses of π^+, π^0 and π^- are very close and they are considered to be three manifestations of the same entity, the pion. This suggests that these particles are degenerate without electromagnetic interactions and the isospin symmetry is a good symmetry under strong interactions. In fact, nuclear interactions between nucleons and pions have an $SU(2)$ isospin symmetry which works very well. The isospin symmetry is very effective to understand the structure of nuclei and nuclear interaction dynamics.

Mathematically the isospin symmetry is just the copy of the spin symmetry; the generators $I^i = \frac{\tau^i}{2}$ $(i = 1, 2, 3)$ satisfy the same $SU(2)$ algebra as the ones of the spin

$$[I^i, I^j] = i\varepsilon_{ijk} I^k. \tag{E.2}$$

where ε_{ijk} is the totally antisymmetric Levi–Cività symbol with $\varepsilon_{123} = 1$ and the summation over k is implied. τ^i have the same expression as the Pauli matrices σ^i $(i = 1, 2, 3)$ and their explicit expression are presented in (C.3).

For the two–nucleon system, analogously to the spin composition of two spin $\frac{1}{2}$, one can construct the $I = 1$(triplet) and $I = 0$(singlet) states of the nucleon–nucleon system,

$$I \;=\; 1(\text{triplet}) \begin{cases} |I = 1, I_3 = 1\rangle = pp, \\ |I = 1, I_3 = 0\rangle = \frac{1}{\sqrt{2}}(pn + np), \\ |I = 1, I_3 = -1\rangle = nn, \end{cases} \tag{E.3}$$

$$I \;=\; 0(\text{singlet}) \quad |I = 0, I_3 = 0\rangle = \frac{1}{\sqrt{2}}(pn - np). \tag{E.4}$$

As can be seen here, $I = 1$(triplet) and $I = 0$(singlet) states are symmetric and antisymmetric, respectively, for interchange of the 1st and 2nd nucleon in the two–nucleon system. One can illustrate this situation in terms of the so-called Young tableaux, in which a doublet state **2**, i.e. the fundamental representation of $SU(2)$ is represented as $\square = \mathbf{2}$.

$$\begin{aligned} \square \otimes \square &= \begin{smallmatrix}\square\\\square\end{smallmatrix} \oplus \square\square \\ \mathbf{2} \otimes \mathbf{2} &= \mathbf{1} \oplus \mathbf{3} \end{aligned} \tag{E.5}$$

where $\begin{smallmatrix}\square\\\square\end{smallmatrix} = \mathbf{1}$(singlet) and $\square\square = \mathbf{3}$(triplet) represent the antisymmetric and symmetric states, respectively. The number below the Young tableaux of (E.5) denotes the dimension of each irreducible representation.

One should be careful in constructing multiplets of nucleon–antinucleon systems. To see this, consider a rotation with π around the 2nd axis in an isospin frame, as an example of isospin transformation of a nucleon doublet,

$$\begin{pmatrix} p' \\ n' \end{pmatrix} = e^{-i\pi\frac{\tau^2}{2}} \begin{pmatrix} p \\ n \end{pmatrix} = -i\tau^2 \begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix}. \tag{E.6}$$

Now, let us define the charge conjugation operator C which transforms particles into antiparticles,

$$C|p\rangle = |\bar{p}\rangle \quad \text{and} \quad C|n\rangle = \bar{n}\rangle, \tag{E.7}$$

where the charge of $\bar{p}$ and $\bar{n}$ is -1 and 0, respectively. Applying C to (E.6) leads to

$$\begin{pmatrix} \bar{p}' \\ \bar{n}' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{p} \\ \bar{n} \end{pmatrix}. \tag{E.8}$$

However, the sign of the eigenvalue of I^3 for antiparticles is opposite to the one for particles and we want to arrange $\bar{n}$ to the upper component ($I^3 = +\frac{1}{2}$) and $\bar{p}$ to the lower component ($I^3 = -\frac{1}{2}$) in the antiparticle doublet, as $\mathbf{2}^* = \begin{pmatrix} \bar{n} \\ \bar{p} \end{pmatrix}$. Then, if we introduce a minus sign to $\bar{n}$, we can see that it transforms in exactly the same way as the the nucleon doublet $\begin{pmatrix} p \\ n \end{pmatrix}$ under the $SU(2)$ isospin transformation, as shown

$$\begin{pmatrix} -\bar{n}' \\ \bar{p}' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\bar{n} \\ \bar{p} \end{pmatrix}. \tag{E.9}$$

(E.9) is certainly the same transformation as (E.6). Namely, $\mathbf{2}^*$ and $\mathbf{2}$ transform in the same way under the $SU(2)$ transformation, i.e. $\mathbf{2}^* \simeq \mathbf{2}$. (Here don't worry about the nasty minus sign to $\bar{n}$.) This can be done only for the $SU(2)$ symmetry as proved in Appendix C.

Now, we can construct the isospin states of the nucleon–antinucleon system as follows;

$$I \;=\; 1(\text{triplet}) \begin{cases} |I=1, I_3=1\rangle = -p\bar{n}, \\ |I=1, I_3=0\rangle = \frac{1}{\sqrt{2}}(p\bar{p} - n\bar{n}), \\ |I=1, I_3=-1\rangle = n\bar{p}, \end{cases} \tag{E.10}$$

$$I \;=\; 0(\text{singlet}) \quad |I=0, I_3=0\rangle = \frac{1}{\sqrt{2}}(p\bar{p} + n\bar{n}). \tag{E.11}$$

E.2 Quark model and $SU(3)$ symmetry

In 1950s, new hadrons were discovered. Those new hadrons were surprisingly long-lived compared to the strong interaction scale. For example, the Λ^0 and K^0 are easily produced in high energy $\pi^- p$ scattering but those particles decay into light hadrons with very long lifetime,

$$\pi^- \;\; + \;\; p \to \Lambda^0 + K^0, \tag{E.12}$$
$$\Lambda^0 \;\; \to \;\; p + \pi^-, \;\; K^0 \to \pi^+\pi^-. \tag{E.13}$$

To explain the fact that while the production of Λ^0 and K^0 occurs with strong interaction scale, the decay of those particles does with weak interaction scale, Nakano and Nishijima and, independently, Gell-Mann introduced a new additive quantum number called "strangeness"(S). They assigned S as $S = 0$ for p, π^-, $S = +1$ for K^0 and $S = -1$ for Λ^0, and considered that while the strong interaction conserves the quantum number S, the weak interaction does not. In the production process (E.12), S is conserved but in the decay processes (E.13), it is not. Soon later, the idea of Nakano, Nishijima and Gell-Mann was confirmed from observed properties of many strange particles discovered those days. The conservation of the strangeness S is similar to the one of the charge Q due to the $U(1)$ electromagnetic symmetry. This suggests the existence of a new $U(1)$ symmetry. Actually one can introduce the symmetry called the $U(1)$ hypercharge symmetry, where the new quantum number Y, called hypercharge, is defined by the sum of the baryon number B and the strangeness S, $Y = B + S$. Because of this new $U(1)$ symmetry, the strong interaction conserves the hypercharge Y and hence, the strangeness S is also conserved in strong interactions, because the baryon number B is a good quantum number for the strong interaction. Then, we can see that the following relation, being called the Nakano–Nishijima–Gell-Mann (NNG) relation (Nakano and Nishijima, 1953; Gell-Mann, 1953),

$$Q = I^3 + \frac{Y}{2}, \tag{E.14}$$

works well for all hadrons discovered those days, where Q and I^3 are the charge and the 3rd component of the isospin of the hadron, respectively.

In 1964, Gell-Mann and Zweig introduced the quarks as physical substances to realize the relation (E.14). In the quark model, all hadrons are made of a few quarks; while all baryons are made of 3 quarks q, all mesons are made of a quark q and an antiquark $\bar{q}$, where all quantum numbers of $\bar{q}$ is opposite to those of q. Since the quark model should make even strange hadrons like Λ^0 and K^0, we need a new quark, i.e. the s(strange) quark in addition to the u(up) and d(down) quarks which nicely build the non-strange hadrons like p, n, π, etc. Thus, in the original quark model the u, d and s quarks were considered to be the fundamental constituents of hadrons and to have the $SU(3)$ symmetry. Note that though the $SU(2)$ symmetry of isospin is a rather good symmetry due to almost equal masses of p and n, the $SU(3)$ symmetry is not such a good symmetry because mass differences of strange hadrons and non-strange hadrons are rather big. Later, the existence of more heavier quarks c(charm), b(bottom) and t(top) quarks were also established. Now, we have 6

	I	I^3	S	B	Y	Q
u	$\frac{1}{2}$	$+\frac{1}{2}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$+\frac{2}{3}$
d	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$
s	0	0	-1	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$

Table E.1 Quantum numbers of u, d, s quarks.

different kinds of quarks q_i $(i = u, d, s, c, b, t)$ and these degree of freedom is called "flavor", which is unrelated to another degree of freedom "color", being the strong interaction charge which plays an important role in quantum chromodynamics, the field theory of strong interactions, being discussed in Chapter 5. The quantum numbers of u, d, s quarks are given in Table E.1. By taking account of these quantum numbers of quarks, we can make many hadrons from those quarks, like $p = (uud)$, $n = (udd)$, $\Lambda^0 = (uds)$, $\pi^+ = (u\bar{d})$, $\pi^- = (\bar{u}d)$, $K^0 = (d\bar{s})$, etc.

In the $SU(3)$ symmetric world,* the fundamental representation of the quark is given by the triplet (=**3**)

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}. \tag{E.15}$$

Now, let us consider the $SU(3)$ transformation of this triplet,

$$\begin{aligned} q_a \to q'_a &= U(\theta)_a^b q_b, \quad (a, b = u, d, s) \\ U(\theta) &= e^{-i\theta^i \frac{\lambda^i}{2}}, \end{aligned} \tag{E.16}$$

where θ^i $(i = 1, 2, \cdots, 8)$ are parameters of this group and $\frac{\lambda^i}{2}$ $(i = 1, 2, \cdots, 8)$ are the generators of the $SU(3)$ group. Here the summation over i is implied. Explicit expression of λ^i is given by (C.4). These generators satisfy the Lie algebra;

$$\left[\frac{\lambda^i}{2}, \frac{\lambda^j}{2}\right] = i f_{ijk} \frac{\lambda^k}{2}, \tag{E.17}$$

where f_{ijk} is called the structure constant and anti-symmetric under any exchange of i, j and k. Here the summation over k is implied. The group $SU(3)$ includes the $SU(2)$ of isospin and the $U(1)$ of hypercharge as subgroups and hence it is the group of rank 2. It has 2 diagonal matrices, λ^3 and λ^8. It is convenient to define the isospin and hypercharge operators as $I^3 = \frac{\lambda^3}{2}$ and $Y = \frac{\lambda^8}{\sqrt{3}}$, respectively. Then,

*Here we consider an idealized world of equal quark masses of u, d and s, though they are, in fact, different. Therefore, the same mathematics can be applied for the color $SU(3)$, which is an exact symmetry.

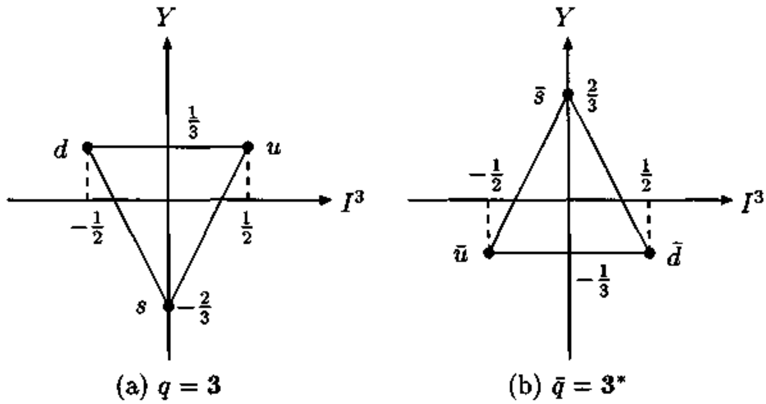


Fig. E.1 Weight diagram for $q = \mathbf{3}$ and $\bar{q} = \mathbf{3}^*$.

we can plot the members of the quark triplet with their quantum numbers of I^3 and Y in (I^3, Y) space, as shown in Fig. E.1(a). The members of the antiquark triplet ($=\mathbf{3}^*$) are also ploted in Fig. E.1(b). (Fig. E.1 is called the weight diagram.) Introducing I^3 and Y into (E.14), we obtain the charge operator,

$$Q = I^3 + \frac{Y}{2} = \frac{\lambda^3}{2} + \frac{\lambda^8}{2\sqrt{3}} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}. \tag{E.18}$$

The diagonal elements of Q are certainly equal to the charge of the u, d and s quarks given in Table E.1. Then we see that the NNG relation works well even for quarks.

E.3 Representations of mesons and baryons

In the quark model, mesons are composed of $q\bar{q}$, while baryons are of qqq. The fundamental representation $q = \mathbf{3}$ transforms as (E.16) under U, while the conjugate representation $\bar{q} = \mathbf{3}^*$ transforms as

$$q^a \to q'^a = q^b (U^\dagger)^a_b \tag{E.19}$$

under $U^\dagger$, where U is unitary, $U^\dagger = U^{-1}$. (see (C.16)) Therefore, it is interesting to see the product of representations to discuss the spectroscopy of mesons and baryons. In general, the product of representations is not irreducible. If some reresentations of the group can be decomposed into a direct sum of other independent representations, they are called "reducible" and if not, they are called "irreducible".

First consider the product $q_a q^b = \mathbf{3} \otimes \mathbf{3}^*$, which is reducible. When we decompose it as

$$q_a q^b = (q_a q^b - \frac{1}{3}\delta_a^b q_c q^c) + \frac{1}{3}\delta_a^b q_c q^c, \tag{E.20}$$

we see that the 1st term is traceless because of $\mathrm{Tr}(q_a q^b) = q_a q^a$ and has 8 components. The 2nd term is just the trace term and of only one component. The 1st and 2nd terms do not mix each other and hence they are irreducible. Then we can say that $\mathbf{3} \otimes \mathbf{3}^*$ can be resolved into the octet $\mathbf{8}$ and singlet $\mathbf{1}$ as

$$\mathbf{3} \otimes \mathbf{3}^* = \mathbf{8} \oplus \mathbf{1}. \tag{E.21}$$

Next, consider the direct product $q_a q_b = \mathbf{3} \otimes \mathbf{3}$. It is also reducible. To resolve this reducible representation into two irreducible representations, let us separate it into the symmetric and antisymmetric parts as

$$q_a q_b = \frac{1}{2}(q_a q_b + q_b q_a) + \frac{1}{2}(q_a q_b - q_b q_a), \tag{E.22}$$

where the 1st and 2nd terms are symmetric and antisymmetric, respectively, for interchange of q_a and q_b. The antisymmetric part has 3 components and identical to the set of objects $\varepsilon_{abc} q_b q_c$, where ε_{abc} is the totally antisymmetric Levi–Cività symbol with $\varepsilon_{123} = 1$. Then if we define $\varepsilon_{abc} q_b q_c \equiv q^a$, we can see that $q_a q^a$ is invariant under the transformation of U as follows;

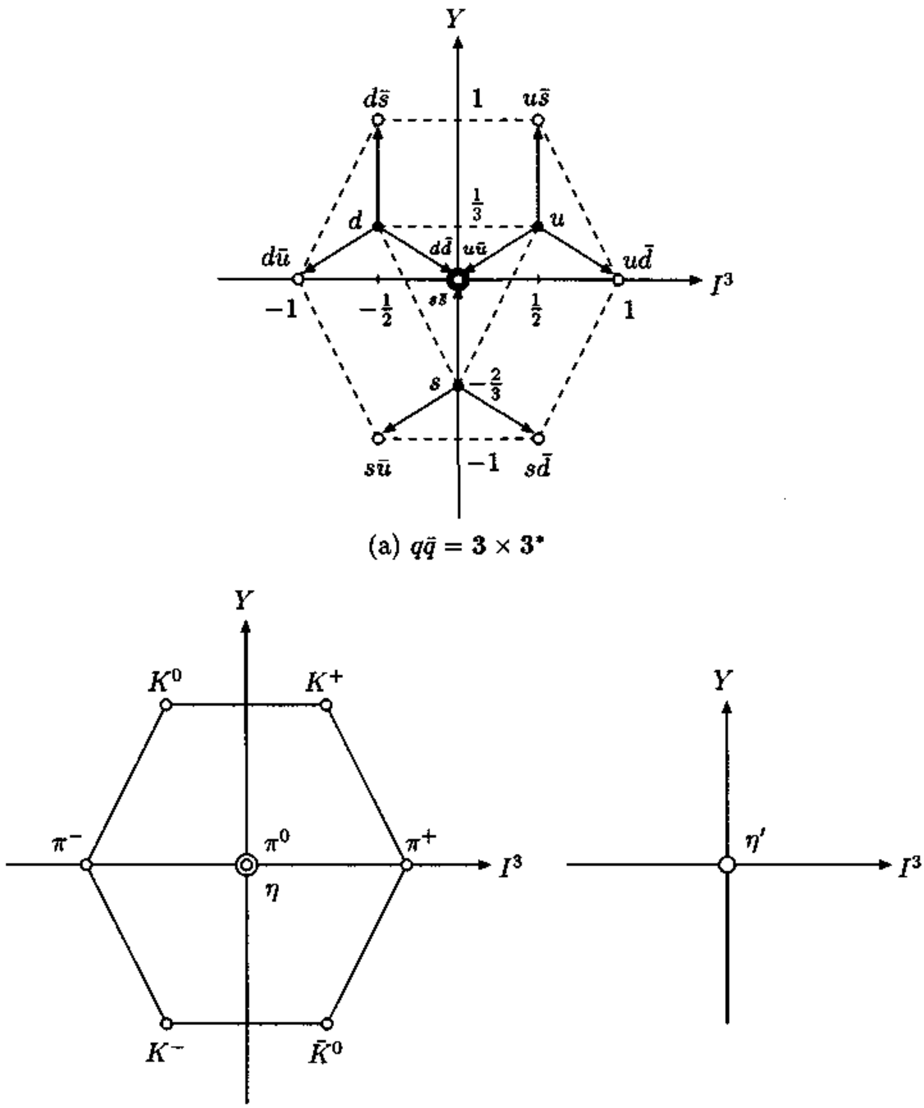
$$\begin{aligned} q'_a q'^a = \varepsilon_{abc} q'_a q'_b q'_c &= \varepsilon_{abc} U_a^{a'} U_b^{b'} U_c^{c'} q_{a'} q_{b'} q_{c'} \\ &= \varepsilon_{a'b'c'} \det U q_{a'} q_{b'} q_{c'} = \varepsilon_{a'b'c'} q_{a'} q_{b'} q_{c'} \\ &= q_{a'} q^{a'} = q_a q^a, \end{aligned} \tag{E.23}$$

where $\det U = 1$. Since U is unitary, from the transformation rule of (E.16) for q_a and the result of (E.23), we can easily find that q^a, i.e. the antisymmetric part of (E.22), transforms as the conjugate representation $\mathbf{3}^*$. The symmetric part has 6 components and thus, we can obtain the decomposition rule,

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}^* \oplus \mathbf{6}. \tag{E.24}$$

$\mathbf{3}^*$ and $\mathbf{6}$ do not mix under the $SU(3)$ transformation and each one cannot be decomposed any more. They are irreducible.

Finally, what happens when we multiply one more quark state to (E.24)? In this case, we have two products $\mathbf{3}^* \otimes \mathbf{3}$ and $\mathbf{6} \otimes \mathbf{3}$. The product $\mathbf{3}^* \otimes \mathbf{3}$ has 9 components and is decomposed into the octet $\mathbf{8}$ and singlet $\mathbf{1}$ as presented in (E.21). But here we should be careful about that in $\mathbf{1}$ all quarks are totally antisymmetric (as can be seen from $q_a q^a$ in (E.23)) and in $\mathbf{8}$ the first 2 quarks are antisymmetric. The product $\mathbf{6} \otimes \mathbf{3}$ has 18 components and is decomposed into the decuplet $\mathbf{10}$ and octet $\mathbf{8}$. In $\mathbf{10}$ all quarks are totally symmetric, while in $\mathbf{8}$ only first 2 quarks are symmetric.



(b) 0^- octet ($\mathbf{8}$)

(c) 0^- singlet ($\mathbf{1}$)

Fig. E.2 Mesons in the quark model.

Thus, we can summarize the result of the decomposition of qqq states as

$$\begin{aligned} \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} &= (\mathbf{3}^* \oplus \mathbf{6}) \otimes \mathbf{3} = (\mathbf{3}^* \otimes \mathbf{3}) \oplus (\mathbf{6} \otimes \mathbf{3}) \\ &= \mathbf{1} \oplus \mathbf{8}_A \oplus \mathbf{8}_S \oplus \mathbf{10}, \end{aligned} \tag{E.25}$$

where the first 2 quarks are antisymmetric and symmetric in $\mathbf{8}_A$ and $\mathbf{8}_S$, respectively.

Likewise in the $SU(2)$ case, we can illustrate the quark triplet (E.15) as $\square = \mathbf{3}$.

In the Young tableaux, $\ydiagram{1,1}$ represents the antisymmetric combination of 2 particles. Since the antisymmetric part of (E.22) being identical to the set of objects $\varepsilon_{abc}q_b q_c \equiv q^a$ has 3 components, we illustrate it as $\ydiagram{1,1} = \mathbf{3}^*$, which is the conjugate representation corresponding to the antiquark $\bar{q}$. Moreover, $\ydiagram{2}$ represents the symmetric combination of 2 particles and thus, the symmetric part of (E.22) having 6 components are illustrated as $\ydiagram{2} = \mathbf{6}$.

- **Mesons**

In the quark model, mesons are made up from a quark q and an antiquark $\bar{q}$. Here we consider an example of pseudoscalar mesons with $J^P = 0^-$.

$$q\bar{q} = \mathbf{3} \otimes \mathbf{3}^* = \mathbf{1} \oplus \mathbf{8}. \tag{E.26}$$

Or using Young tableaux, one can write this as

$$\begin{array}{ccc} q\bar{q} = \ydiagram{1} \otimes \ydiagram{1,1} & = & \ydiagram{1,1,1} \oplus \ydiagram{2,1}. \\ \mathbf{3} \otimes \mathbf{3}^* & = & \mathbf{1} \oplus \mathbf{8} \end{array} \tag{E.27}$$

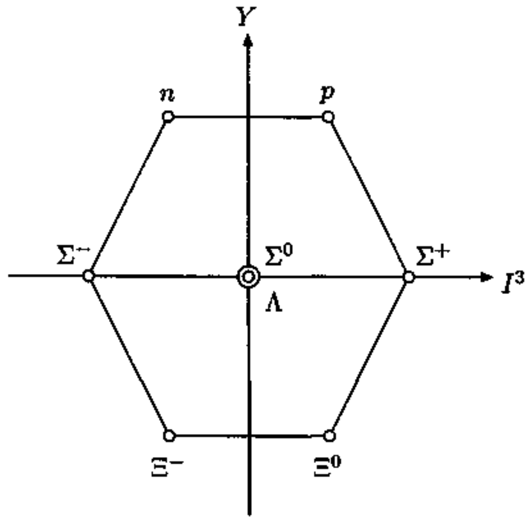
The number below the Young tableaux denotes the dimension of each irreducible representation. Therefore, mesons are represented by a sum of the octet $\mathbf{8}$ and the singlet $\mathbf{1}$. By combining the weight diagrams of q (Fig. E.1(a)) and $\bar{q}$ (Fig. E.1(b)), we can show this in a different way as shown in Fig. E.2. The same configuration is obtained even for vector mesons with $J^P = 1^-$.

- **Baryons**

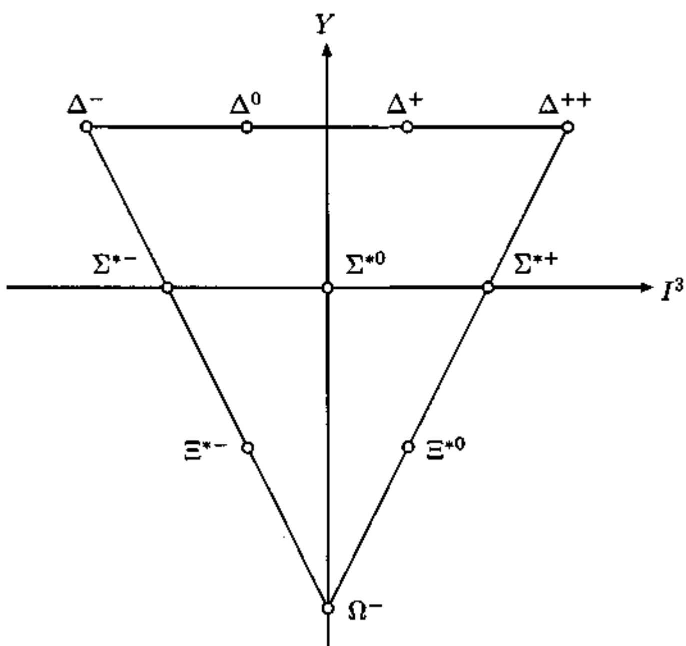
Baryons are composed of three quarks qqq and its product of representations are given in (E.25). The Young tableaux for this configuration is given as

$$\begin{array}{ccl} qqq & = & \ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} = \left(\ydiagram{1,1} \oplus \ydiagram{2}\right) \otimes \ydiagram{1} = \ydiagram{1,1,1} \oplus \ydiagram{2,1} \oplus \ydiagram{2,1} \oplus \ydiagram{3} \\ & & \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (\mathbf{3}^* \oplus \mathbf{6}) \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \end{array} \tag{E.28}$$

Similarly to mesons, by combining the weight diagram of Fig. E.1(a), we can represent the baryons as in Fig. E.3.



(a) $J^P = \frac{1}{2}^+$ baryon octet (**8**)



(b) $J^P = \frac{3}{2}^+$ baryon decuplet (**10**)

Fig. E.3 Baryons in the quark model.

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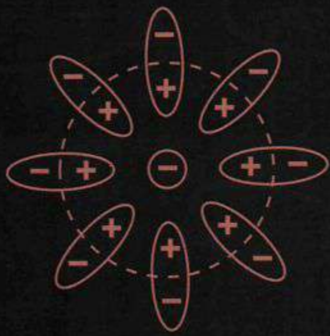
Index



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