NOTE: To extract the LaTeX source and other sources files from this PDF file, execute:

```
pdftk curve_intersection_potpourri.pdf unpack_files output .
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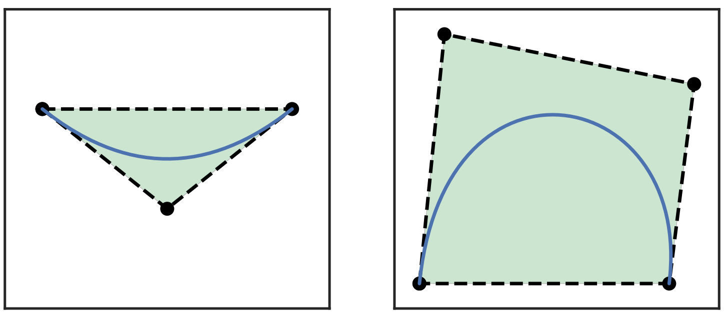
Contents

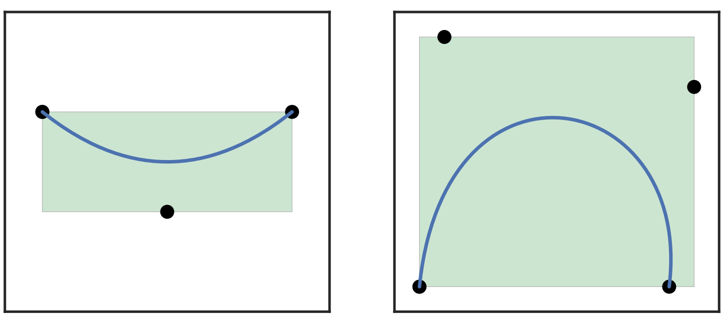
1 Some Bézier Facts

1.1 Bounding Box

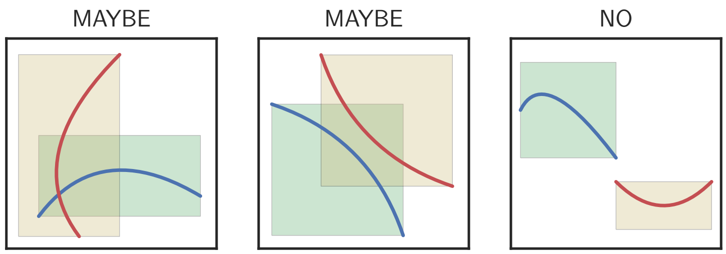
For a Bézier curve, de Casteljau's algorithm (cite Ch. 4 Farin2001) ensures that the graph of $\gamma([0, 1])$ is contained in the convex hull of the control points:



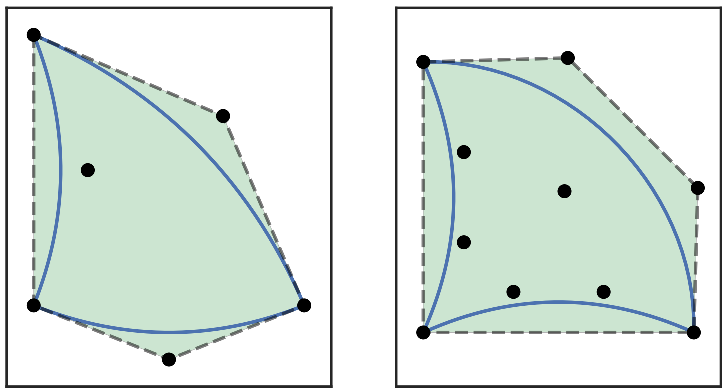
As an easier to compute shape (and easier to compute with), we instead use bounding rectangles which simply find the extremal x and y values among the control points:



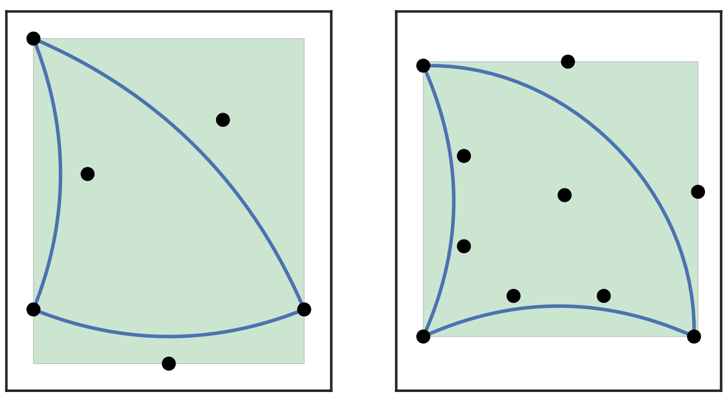
This way, a first check that curves **don't** intersect can be done quickly:



Similarly, for a Bézier triangle (cite Ch. 17 Farin2001), the triangular[1] de Casteljau algorithm also guarantees the graph of $\gamma(T)$[2] is contained in the convex hull of the control points.



but we still prefer the bounding rectangle for ease of computation:



2 Curve-Curve Intersection

This is a collection of things that could (should?) be said somewhere, but not in the main paper.

2.1 Intersection Corner Cases

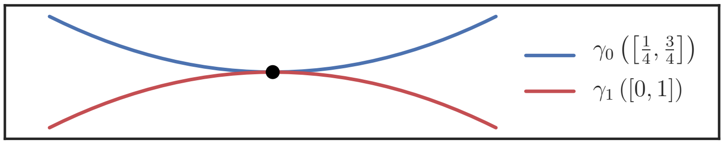
Via implicitization, algebraic curve intersection is equivalent to polynomial root finding. Just as with polynomial root finding, non-simple roots can cause loss of numerical precision.

[1]Using three barycentric coordinates in $\mathbf{R}^2$.
[2]Here T is the unit simplex in $\mathbf{R}^2$.

2.1.1 Double Root

For example, two tangent curves correspond to a double (or worse) root. To see this, consider

$$\gamma_0(s) = \frac{1}{2}\begin{bmatrix} 4s \\ (2s-1)^2 \end{bmatrix}, \quad \gamma_1(t) = \frac{1}{8}\begin{bmatrix} 8t+4 \\ -(2t-1)^2 \end{bmatrix}. \tag{1}$$



These curves are tangent at $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ when $s = t = \frac{1}{2}$. Implicitizing γ_0 and plugging in γ_1 gives

$$f_0(x, y) = -2\left(x^2 - 2x - 2y + 1\right) \Longrightarrow 0 = f_0\left(x_1(t), y_1(t)\right) = -(2t-1)^2 \tag{2}$$

which is a double root as expected.

2.1.2 Double Root: Zero Function

If we instead formulate the problem as finding zeros of

$$F(s,t) = \gamma_0(s) - \gamma_1(t). \tag{3}$$

This function has Jacobian

$$DF = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ 8s-4 & 2t-1 \end{bmatrix} \tag{4}$$

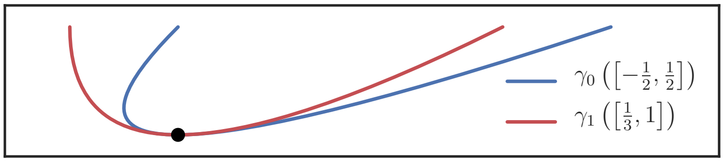
which is not invertible (since there is a zero row) at the solution $s = t = \frac{1}{2}$.

2.1.3 Triple Root

For an even worse example, note that two degree two parametric curves can intersect, be tangent and have the same curvature at a point without being the same curve.

Consider

$$\gamma_0(s) = \begin{bmatrix} s(2s+1) \\ s^2 \end{bmatrix}, \quad \gamma_1(t) = \frac{1}{4}\begin{bmatrix} 3t(3t-2) \\ (3t-2)^2 \end{bmatrix}. \tag{5}$$



Implicitizing γ_0 and plugging in γ_1 gives a triple root:

$$f_0(x, y) = x^2 - 4xy + 4y^2 - y \Longrightarrow 0 = f_0\left(x_1(t), y_1(t)\right) = \frac{3}{16}(t-2)(3t-2)^3. \tag{6}$$

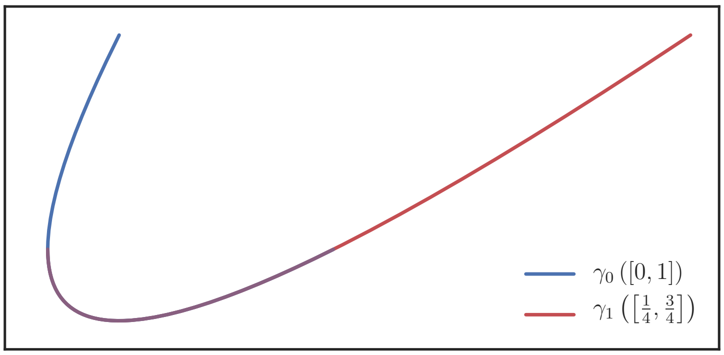
So the curves intersect at $\gamma_0(0) = \gamma_1\left(\frac{2}{3}\right) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and at these points, the tangent vectors are parallel to $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ (hence making the curves tangent). Finally, the (signed) curvature of each of these functions is identical: $\kappa_0(0) = \kappa_1\left(\frac{2}{3}\right) = 2$. However, the curves are not the same:

$$f_1(x, y) = \frac{81}{16}\left(x^2 - 2xy + y^2 - y\right). \tag{7}$$

2.1.4 Same Curve

Another problem arises when we try to intersect two different sections of the same curve (i.e. coincident curves). This is not the same kind of corner case, i.e. there isn't the same worry about numerical loss of precision. Instead, the problem is that we now have infinitely many intersections (a continuum). As an example:

$$\gamma_0(s) = \frac{1}{4} \left[\begin{array}{c} (3s-1)^2 - 1 \\ (3s-2)^2 \end{array} \right], \quad \gamma_1(t) = \frac{1}{16} \left[\begin{array}{c} (12t-5)(12t-1) \\ (12t-5)^2 \end{array} \right]. \tag{8}$$



However, implicitizing, we find both give

$$16 f_0(x, y) = f_1(x, y) = 81\left(x^2 - 2xy + y^2 - y\right). \tag{9}$$

In some sense, we'd like to find the two endpoints that define the **shared section** of the curve. They are

$$\gamma_0\left(\frac{1}{3}\right) = \gamma_1\left(\frac{1}{4}\right) = \frac{1}{4}\left[\begin{array}{c} -1 \\ 1 \end{array} \right] \qquad \gamma_0(1) = \gamma_1\left(\frac{7}{12}\right) = \frac{1}{4}\left[\begin{array}{c} 3 \\ 1 \end{array} \right]. \tag{10}$$