Mathematical modeling and control of nematode impact on banana production

1 Model formulation

Herein, we formulate a mathematical model describes the infestation dynamics of roots of banana by nematodes during the cropping season in a plantation. The model is based on the life cycle of *Radopholus Similis* presented in the previous section. We use the compartmental modeling approach with a progression of stages.

To build the model, we make the additional modelling assumptions described below:

1. The nematode population is divided into two compartments: free nematodes in the soil (N_F) and infesting nematodes in the roots (N_I).

2. The root biomass of banana plants in the plantation is subdivided into two compartments: healthy roots (S) and infected roots (I).

   Moreover, we define by $\mathcal{X} = \frac{N_I}{I}$ the infestation density.

3. Healthy roots are produced at a constant rate (Λ)

4. Free nematodes (N_F) infest the healthy roots (rate β_F) and immediately, we have two different losses:

   i) The lost of healthy roots that become infected roots denoted by $-\beta_F N_F S$

   ii) The lost of free nematodes that become infesting nematodes at the rate $-\alpha\beta_F N_F S$, where α being conversion of the number of nematodes per root unit.

   They undergo natural mortality (rate ν_F).

5. Infesting pests (N_I) feed on the plant roots with a Holling type II-like functional response $dN\dfrac{I}{a+I}$ that is well-suited for invertebrates [?]. They undergo natural mortality (rate ν_I). This mortality rate differs from the mortality rate in the soil because the environments are different. The root, which serves both as host and food for the nematode, is more favourable to pest survival than the soil ($\nu_I < \nu_F$).

6. Infesting nematodes reproduce inside the roots following the logistic function $\left(r + \rho d \frac{I}{a+I}\right) N_I \left(1 - \frac{\mathcal{X}}{K}\right)$, where ρ, K and r are respectively the conversion rate of ingested biomass into pests, the maximum reception capacity of the environment and the intrinsic growth rate of infesting nematodes that does not depend on roots consumption.

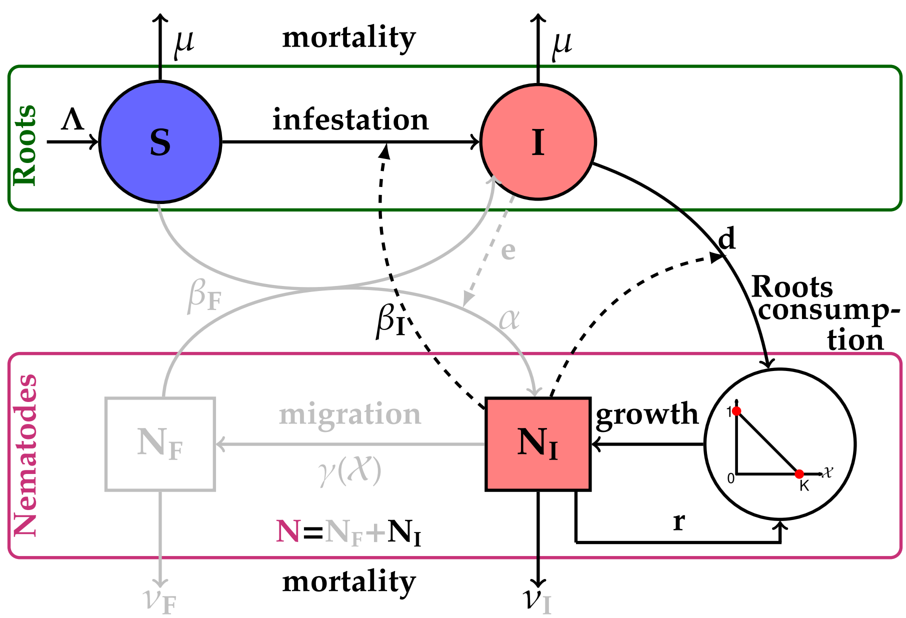
7. Infesting pests (N_I) infest the healthy roots (rate β_I) and immediately, we have the lost of healthy roots that become infected roots denoted by $-\beta_I N_I S$.

8. When the roots no longer present favorable living and feeding conditions, the nematodes abandon it and enter the soil in search of a new root that they infest immediately at the rate $\left(\gamma + \gamma \frac{\mathcal{X}}{K}\right)$. γ is the natural exit rate of infesting nematodes.

9. Free nematodes infest the infected roots (rate β_F) and immediately, we have the lost of free nematodes that become infesting nematodes denoted by $-\alpha \beta_F e N_F I$ and $e < 1$ since healthy roots are more attacked compared to infected roots due to infection.

10. All roots undergo natural mortality (rate μ).

Under the above hypotheses, we construct the compartmental model 1.



From the compartmental model in Figure 1, we derive the following system of nonlinear ordinary differential equations:

$$
\begin{cases}
\frac{dS}{dt} = \Lambda - (\beta_F N_F + \beta_I N_I)S - \mu S \\
\frac{dI}{dt} = (\beta_F N_F + \beta_I N_I)S - \mu I - dN_I \frac{I}{a+I} \\
\frac{dN_F}{dt} = -\alpha\beta_F(S+eI)N_F + \left(\gamma + \gamma\frac{N_I}{KI}\right)N_I - \nu_F N_F \\
\frac{dN_I}{dt} = \alpha\beta_F(S+eI)N_F - \left(\gamma + \gamma\frac{N_I}{KI}\right)N_I + \left(r + \rho d\frac{I}{a+I}\right)N_I\left(1 - \frac{N_I}{KI}\right) - \nu_I N_I
\end{cases}
$$

Full model

⇕

Reduced model

(Tikhonov's theorem)

The variables and parameters used in above model are described in the table below

Table 1: Description of model parameters

Symbol	Meaning	Value	Source
Λ	Recruitment	300 $g.day^{-1}$	Assumed
β_F	Infection rate of N_F	$1 \times 10^{-7} nematode^{-1}.day^{-1}$	Assumed
$\beta_I = \beta$	Infection rate of N_I	$1 \times 10^{-7} nematode^{-1}.day^{-1}$	Assumed
α'	Conversion of the number of nematodes per root unit	100 g^{-1}	ref^1
μ	Mortality rate of root	0.05 day^{-1}	Assumed
d	Consumption rate	$2.10^{-4} g.nematode^{-1}.day^{-1}$	ref^2
a	Half-saturation constant	60 g	ref^2
ρ	Conversion rate of ingested roots	400 $nematode.g^{-1}$	ref^2
r	Intrinsic growth rate of N_I	0.15 day^{-1}	Assumed
K	Maximum capacity	1000 $nematode.g^{-1}$	Assumed
γ	Natural exit rate of N_I	0.02 day^{-1}	Assumed
e	Probability of reinfection	0.0002	Assumed
ν_I	Mortality rate of N_I	0.045 day^{-1}	ref^2
ν_F	Mortality rate of N_F	0.0495 day^{-1}	ref^2

2 Optimal control strategy

2.1 Problem statement

In this section, we extend the above reduced system to include a continuous optimal control problem which consists in maximizing the yield, while minimising the controls costs as well as the infestation for the next cropping season using biostimulants. This control technique has been developed as an environmentally safe bio-insecticide that is sprayed but not toxic to workers and reduces nematodes fertility (infestation rate) when drilling the entry hole in the roots. Denote by $u(t)$ the effort made to reduce the nematode infestation over a time interval $[0, t_f]$ where t_f denotes the end time of cropping season. Hence we obtain the following controlled system:

$$\dot{S} = \Lambda - (1-u)\beta NS - \mu S \tag{1a}$$

$$\dot{I} = (1-u)\beta NS - \mu I - dN\frac{I}{a+I} \tag{1b}$$

$$\dot{N} = \left(r + \rho d\frac{I}{a+I}\right)N\left(1 - \frac{N}{KI}\right) - \mu_i N \tag{1c}$$

The set of admissible controls is defined as follows

$$\mathcal{U} = \{u \in L^\infty([0,t_f]) : 0 \le u \le u_{max}, \forall t \in [0,t_f]\} \tag{2}$$

The optimal control problem is then formulated as follows.

Problem 1. *Find an admissible control* $u^* \in \mathcal{U}$ *and the corresponding state variable* $x^* = (S^*, I^*, N^*)$ *minimizing the following objective functional:*

$$\mathcal{J}(u) = \int_0^{t_f} L(x(t),u(t))dt + \psi(x(t_f)) \tag{3}$$

with

$$L(x(t),u(t)) = B_u u(t) - B_S S(t) \qquad \text{and} \qquad \psi(x(t_f)) = B_I I(t_f)$$

and x(t) solution of system (1)

This type of problem is called a Bolza optimal control. The first term, i.e. the integral, represents the production losses over the entire cultivation period, with B_S the control weight constant of healthy roots. The second represents the infected roots which remain after harvest in the plantation, weighted by the parameter B_I. Finally, $\mathcal{J}$ represents the production losses over the entire cultivation period, penalized by the infected final roots.

Let us consider the vector $x = (S, I, N)^T$. In vector notation the system (1) takes the form

$$\dot{x} = f(x) + ug(x)$$

with

$$\mathbf{f(x)} = \begin{pmatrix} \Lambda - \beta NS \\ \beta NS - \mu I - dN\frac{I}{a+I} \\ \left(r + \rho d\frac{I}{a+I}\right)N\left(1 - \frac{N}{KI}\right) - \mu_i N \end{pmatrix} \qquad \text{and} \qquad \mathbf{g(x)} = \begin{pmatrix} \beta NS \\ -\beta NS \\ 0 \end{pmatrix}.$$

2.2 Necessary optimal conditions

The following theorem establish the existence of an optimal control for the problem (1).

Theorem 1. *There exists an optimal control* $u^* \in \mathcal{U}$ *and a corresponding solution* $x^* = (S^*, I^*, N^*)$ *of the problem* (1) *that minimizes the cost function* $\mathcal{J}$ *such that* $\mathcal{J}(u^*) = \underset{0 \leq u(t) \leq u_{max}}{min} \mathcal{J}(u)$.

Proof. Fleming and Rishel [1] imposes three conditions:

1. The set of controls and corresponding state variables is non-empty.
2. The control set $\mathcal{U}$ is closed and convex.
3. The integrand L of the objective functional is convex on $\mathcal{U}$ and there exist non negatif constants $c_1 > 0$, c_2 and $\eta > 1$ satisfying $L(x(t), u(t)) \geq c_1 |u|^{\eta/2} - c_2$.

Firstly, an existence result in Lukes ([2]; Theorem 9.2.1) for the controlled system (1) with bounded coefficients on the finite interval time is used to give the first condition. Secondly, let us consider the functional

$$\phi_t : L^\infty([0,T]) \to [0, u_{max}]$$
$$u(.) \mapsto u(t),$$

we have $|\phi_t(u)| = |u(t)| \leq u_{max}$, $\forall w \in L^\infty([0,T])$. Therefore ϕ_t is linear continuous. From this continuity, we deduce that $\phi_t^{-1}([0, u_{max}])$ is closed and then $\mathcal{U}$ is closed. Also for $(u_1, u_1) \in \Gamma$ and $\alpha \in [0,1]$, we have :

$$\forall t \in [0,T],\ \alpha u_1(t) + (1-\alpha) u_2(t) \leq \alpha u_{max} + (1-\alpha) u_{max} = u_{max}.$$

We deduce that

$$\alpha u_1 + (1-\alpha) u_2 \in \mathcal{U}$$

and then $\mathcal{U}$ is convex. So the second condition is satisfied. For the third condition, it is easy to see that the integrand function L of the objective functional is convex in the controls since it is linear. Also, for $c_1 = B_u$, and $\eta = 2$, there exist $c_2 = B_S \mathcal{A}$ such that

$$L(u(t), S(t)) = B_u u - B_S S \geq c_1 |u|^{\eta/2} - c_2$$

□

The existence of the optimal control being established, we now apply the Pontryagin's Maximum Principle [3] to establish the first-order necessary optimality conditions. This optimality conditions is use to compute the optimal control u^* of problem (1). This principle transforms the optimization problem into a problem of determining the minimum with respect to u^* of the Hamiltonian of problem (1). Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ the adjoint variables, $\lambda^0 \in \mathbb{R}$ and $\langle . \rangle$ the scalar product in $\mathbb{R}^3$. The Hamiltonian of problem

(1) is define as follows

$$
\begin{aligned}
\mathcal{H}(x,u,\lambda,\lambda^0) =& \lambda^0[B_u u - B_S S] + \langle \lambda, f(x) + ug(x)\rangle \\
=& \lambda^0[B_u u - B_S S] + \lambda_1[\Lambda - (1-u)\beta NS - \mu S] \\
&+ \lambda_2\left[(1-u)\beta NS - \mu I - dN\frac{I}{a+I}\right] \\
&+ \lambda_3\left[\left(r + \rho d\frac{I}{a+I}\right)N\left(1-\frac{N}{KI}\right) - \mu_i N\right].
\end{aligned} \tag{4}
$$

According to the Pontryagin's Maximum Principle,we obtain the following theorem

Theorem 2. *Let $u^*(.)$ be an optimal control and $x^* = (S^*, I^*, N^*)$ the associated trajectory on $[0, t_f]$ starting from x_0. Then we have the following conditions:*

- *There exists an absolutely continuous vector mapping $\lambda : [0, t_f] \to \mathbb{R}^3$ called the adjoint vector, which satisfies the following adjoint equation $\dot{\lambda} = \dfrac{\partial \mathcal{H}}{\partial x}(x(t), u(t), \lambda(t), \lambda^0)$ $\forall t \in [0, t_f]$, i.e:*

$$
\begin{cases}
\dot{\lambda}_1 = B_S + (1-u)\beta N(\lambda_1 - \lambda_2) + \mu\lambda_1 \\
\dot{\lambda}_2 = \left(\mu + da\frac{N}{(a+I)^2}\right)\lambda_2 - \left(\rho da\frac{N}{(a+I)^2}\left(1-\frac{N}{KI}\right) + \frac{1}{K}\left(\frac{N}{I}\right)^2\left(r + \rho d\frac{I}{a+I}\right)\right)\lambda_3 \\
\dot{\lambda}_3 = (1-u)\beta S(\lambda_1 - \lambda_2) + d\frac{I}{a+I}\lambda_2 - \left(\left(r + \rho d\frac{I}{a+I}\right)\left(1 - 2\frac{N}{KI}\right) - \mu_i\right)\lambda_3
\end{cases} \tag{5}
$$

- *There exists $\lambda^0 \geq 0$ such that the vector $(\lambda(.), \lambda^0)$ is non trivial i.e $(\lambda(.), \lambda^0) \neq (0,0,0,0)$.*

- *The transversality conditions satisfies $\lambda(t_f) = \nabla\psi(x(T))\lambda^0$, that is*

$$
\lambda_1(t_f) = \lambda_3(t_f) = 0 \qquad \text{and} \qquad \lambda_2(t_f) = B_2\lambda^0. \tag{6}
$$

- *The control $u(.)$ satisfies the minimisation condition:*

$$
u^*(t) \in \arg\min_{\omega \in \mathcal{U}} \mathcal{H}(x(t), \omega(t), \lambda(t), \lambda^0), \quad \forall t \in [0, t_f] \tag{7}
$$

Remark 1.
- *An extremal trajectory of the optimal problem (1) is defined as a quadruplet $(x(.), u(.), \lambda(.), \lambda^0)$ satisfying the state equation (1) and equations (4)-(7).*
- *An extremal trajectory of the optimal problem (1) does not necessarily correspond to an optimal trajectory (the PMP is only a first-order necessary condition).*

If $\lambda^0 = 0$, then we have $\lambda(t_f) = 0$, which contradicts the third condition of theorem (2). Hence $\lambda^0 \neq 0$, and we choose $\lambda^0 = 1$. Now, as $\mathcal{H}$ is linear of u, its minimization then depends on the sign of

$$
\partial_u \mathcal{H}(x(t), \omega(t), \lambda(t)) = B_u - \beta NS(\lambda_2 - \lambda_1)
$$

Hence, by introduce the switching function

$$\phi(t) = B_u - \beta NS(\lambda_2 - \lambda_1),$$

we characterise the optimal control u^* as follows

$$u^*(t) = \begin{cases} 0 & \text{if} \quad \phi(t) > 0 \\ undefined & \text{if} \quad \phi(t) = 0 \\ u_{max} & \text{if} \quad \phi(t) < 0 \end{cases}$$

2.3 Singular control

A control is called singular if there exists a nonempty interval $[t_1, t_2] \subset [0, t_f]$ such that $\phi(t) = 0$ for all$t \in [t_1, t_2]$.

Theorem 3. *If there exists a singular control $u^*(t)$ on $[t_1, t_2]$, then it is order 1 and locally minimizing . Its expression is given by*

$$u^*_{sing}(t) = 1 + \frac{\mathcal{D}(x,\lambda)}{\mathcal{F}(x,\lambda)},$$

where

$$\begin{aligned}
\mathcal{D}(x,\lambda) =& \frac{\rho da}{(a+I)^2}\left[\mu I + dN\frac{I}{a+I}\right]\left(1 - \frac{N}{KI}\right) + \frac{1}{KI^2}\left[I\frac{dN}{dt} + \mu NI + dN^2\frac{I}{a+I}\right]\left(r + \rho d\frac{I}{a+I}\right) + \frac{\Lambda}{S^2}(\Lambda - \mu S) \\
& - \beta\left[S\frac{dN}{dt} + (\Lambda - \mu S)N\right]\mathcal{Y} - \frac{da}{(a+I)^3}\beta NS\left[(a+I)\frac{dN}{dt} + 2\left(\mu I + dN\frac{I}{a+I}\right)N\right]\lambda_2 \\
& - \frac{da}{(a+I)^2}\left[-\left(\frac{dN}{dt} + \frac{\Lambda N}{S}\right) + \beta N^2 S(\mu\lambda_2 + B_1)\right] \\
& + \left(1 - \frac{N}{KI}\right)\frac{\rho da}{(a+I)^3}\beta NS\left((a+I)\frac{dN}{dt} + 2\left(\mu IN + dN^2\frac{I}{a+I}\right)\right)\lambda_3 \\
& - \frac{\rho da}{(a+I)^2}\frac{1}{KI^2}\beta N^2 S\left(I\frac{dN}{dt} + \mu IN + dN^2\frac{I}{a+I}\right)\lambda_3 \\
& + \frac{\rho da}{(a+I)^2}\left(1 - \frac{N}{KI}\right)\beta N^2 S\left[d\frac{I}{a+I}\lambda_2 - \left(\left(r + \rho d\frac{I}{a+I}\right)\left(1 - \frac{2N}{KI}\right) - \mu_i\right)\lambda_3\right] \\
& + \frac{2}{KI^3}\beta N^2 S\left(I\frac{dN}{dt} + \mu IN + dN^2\frac{I}{a+I}\right)\left(r + \rho d\frac{I}{a+I}\right)\lambda_3 \\
& - \frac{1}{KI^2}\frac{\rho da}{(a+I)^2}\beta N^3 S\left(\mu I + dN\frac{I}{a+I}\right)\lambda_3 \\
& + \frac{1}{KI^2}\beta N^3 S\left(r + \rho d\frac{I}{a+I}\right)\left[d\frac{I}{a+I}\lambda_2 - \left(\left(r + \rho d\frac{I}{a+I}\right)\left(1 - \frac{2N}{KI}\right) - \mu_i\right)\lambda_3\right],
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}(x,\lambda) =& -2\frac{\rho da}{(a+I)^2}\beta NS\left(1 - \frac{N}{KI}\right)\left[1 + \frac{\beta N^2 S\lambda_2}{a+I}\right] - \frac{2}{KI^2}\beta N^2 S\left(r + \rho d\frac{I}{a+I}\right)\left[1 + \frac{\beta N^2 S\lambda_3}{I}\right] \\
& - 2\frac{\Lambda\beta N}{S} - \beta\frac{dN}{dt} + 2\frac{da}{(a+I)^2}\beta^2 N^3 S^2\left[\frac{\lambda_2}{a+I} + \frac{\rho N\lambda_3}{KI^2}\right],
\end{aligned}$$

$$\mathcal{Y} = \left(da\frac{N}{(a+I)^2}\lambda_2 - \left(\rho da\frac{N}{(a+I)^2}\left(1-\frac{N}{KI}\right) + \frac{1}{K}\left(\frac{N}{I}\right)^2\left(r+\rho d\frac{I}{a+I}\right)\right)\lambda_3 - B_1 \right)$$

Proof. Suppose that there exists a singular control u^*_{sing} for all $t \in [t_1, t_2]$. Then we necessarily have $\phi(t) = 0$ for all $t \in [t_1, t_2]$. Using this fact that $\phi(t) = 0$ for all $t \in [t_1, t_2]$, we obtain

$$\lambda_2 - \lambda_1 = \frac{B_u}{\beta NS} \tag{8}$$

Thus, differentiating the switching function with respect to time t along the singular control and the adjoint system (13) and using equation (8), we obtain:

$$\dot{\phi}(t) = -\left(r+\rho d\frac{I}{a+I}\right)\left(1-\frac{N}{KI}\right) + \mu_i - \frac{\Lambda}{S} - \beta NS\mathcal{Y}$$

Now, differentiating ϕ a second time since the expression of its first differential does not explicitly depend on the control, we obtain:

$$\begin{aligned}
\ddot{\phi}(t) = &-\left(r+\rho d\frac{I}{a+I}\right)\left(1-\frac{N}{KI}\right) + \mu_i - \frac{\Lambda}{S} \\
&- \beta NS\left[da\frac{N}{(a+I)^2}\lambda_2 - \left(\rho da\frac{N}{(a+I)^2}\left(1-\frac{N}{KI}\right) + \frac{1}{K}\left(\frac{N}{I}\right)^2\left(r+\rho d\frac{I}{a+I}\right)\right)\lambda_3 - B_1 \right] \\
&+ \frac{\Lambda}{S^2}(\Lambda - (1-u)\beta NS - \mu S) - \beta\left[S\frac{dN}{dt} + (\Lambda - (1-u)\beta NS - \mu S)N \right]\mathcal{Y} \\
&- \frac{da}{(a+I)^3}\beta NS\left[(a+I)\frac{dN}{dt} - 2\left((1-u)\beta NS - \mu I - dN\frac{I}{a+I}\right)N \right]\lambda_2 - da\frac{N}{(a+I)^2}\beta NS(\mathcal{Y} + \mu\lambda_2 + B_1) \\
&+ \left(1-\frac{N}{KI}\right)\frac{\rho da}{(a+I)^3}\beta NS\left((a+I)\frac{dN}{dt} - 2\left((1-u)\beta N^2 S - \mu IN - dN^2\frac{I}{a+I}\right)\right)\lambda_3 \\
&- \rho da\frac{N}{(a+I)^2}\frac{1}{KI^2}\beta NS\left(I\frac{dN}{dt} - (1-u)\beta N^2 S + \mu IN + dN^2\frac{I}{a+I}\right)\lambda_3 \\
&+ \rho da\frac{N}{(a+I)^2}\left(1-\frac{N}{KI}\right)\beta NS\left[(1-u)\beta S(\lambda_1-\lambda_2) + d\frac{I}{a+I}\lambda_2 - \left(\left(r+\rho d\frac{I}{a+I}\right)\left(1-\frac{2N}{KI}\right) - \mu_i\right)\lambda_3 \right] \\
&+ \frac{2}{KI^2}\left(I\frac{dN}{dt} - (1-u)\beta N^2 S + \mu IN + dN^2\frac{I}{a+I}\right)\left(\frac{N}{I}\right)\left(r+\rho d\frac{I}{a+I}\right)\lambda_3\beta NS \\
&+ \frac{1}{K}\left(\frac{N}{I}\right)^2\frac{\rho da}{(a+I)^2}\left((1-u)\beta NS + \mu I + dN\frac{I}{a+I}\right)\lambda_3\beta NS \\
&+ \frac{1}{K}\left(\frac{N}{I}\right)^2\left(r+\rho d\frac{I}{a+I}\right)\left[(1-u)\beta S(\lambda_1-\lambda_2) + d\frac{I}{a+I}\lambda_2 - \left(\left(r+\rho d\frac{I}{a+I}\right)\left(1-\frac{2N}{KI}\right) - \mu_i\right)\lambda_3 \right]\beta NS
\end{aligned}$$

As $\dot{\phi}(t) = 0$ for all $t \in [t_1, t_2]$, we have:

$$\beta NS\mathcal{Y} = -\left(r+\rho d\frac{I}{a+I}\right)\left(1-\frac{N}{KI}\right) + \mu_i - \frac{\Lambda}{S} = -\left(\frac{1}{N}\frac{dN}{dt} + \frac{\Lambda}{S}\right) \tag{9}$$

Then, substituting equation (9) into the above expression of $\ddot{\phi}(t)$, we obtain:

$$\ddot{\phi}(t) = \mathcal{D}(x,\lambda) + (1-u)\mathcal{F}(x,\lambda)$$

Hence, as $\ddot{\phi}(t) = 0$, we obtain the expression of singular control as follows

$$u^*_{sing}(t) = 1 + \frac{\mathcal{D}(x,\lambda)}{\mathcal{F}(x,\lambda)}$$

Since 2 successive derivatives of the switching function made it possible to achieve singular control, then this singular control is of order 1. □

Therefore, the structure of the optimal control u^* is given by:

$$u^*(t) = \begin{cases} 0 & \text{if} \quad \phi(t) > 0 \\ u^*_{sing}(t) & \text{if} \quad \phi(t) = 0 \\ u_{max} & \text{if} \quad \phi(t) < 0 \end{cases}$$

2.4 Numerical solutions

For all numerical simulations, we take $B_S = 2$.

1^{st} **case** In this first case, we take $B_u = 1000$ and $B_I = 5$.

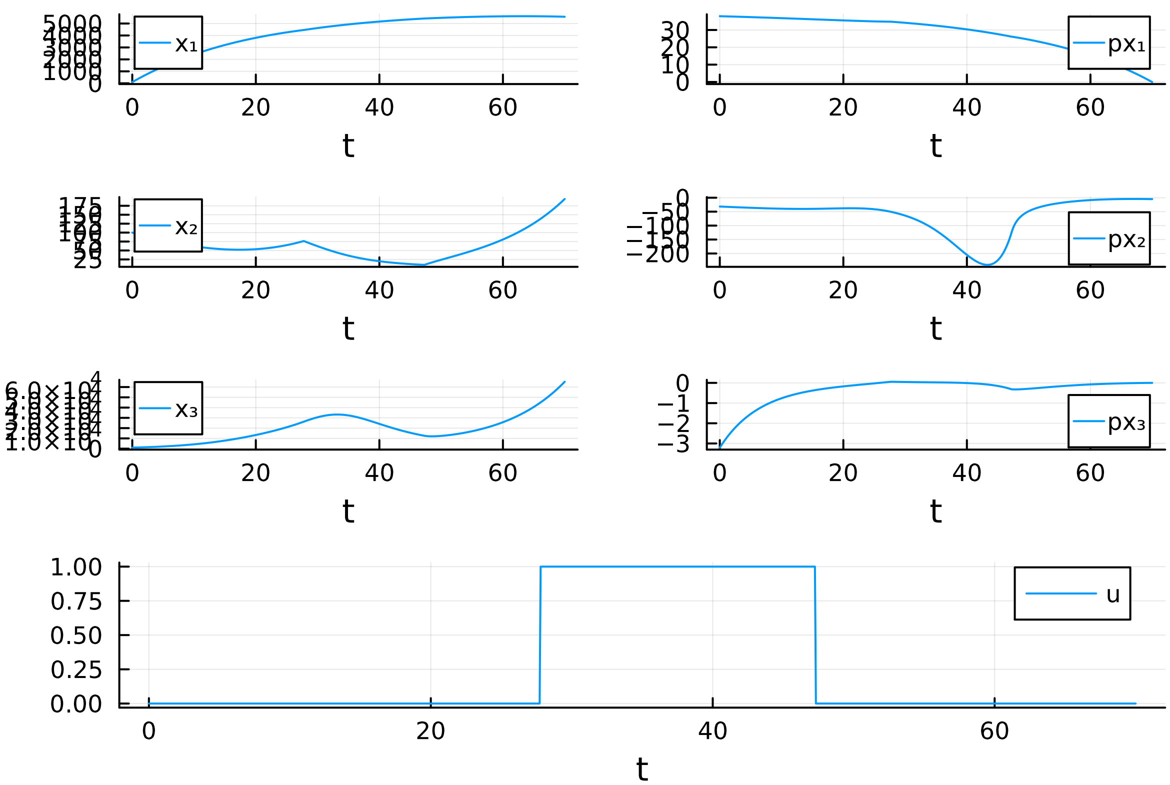


Figure 1: Simulations of system (1) .

References

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