Analytical Track Linearization and 3D Point of Closest Approach

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Contents

1 Abstract

In this report, we derive the analytical Jacobians for particle tracks (1) in the absence of electromagnetic fields and (2) in the presence of a constant B-field in z-direction. Our work is a follow-up to Sec. 5.3.3 and 5.3.4 of Ref. [1]: We (1) simplify the results obtained there by evaluating the Jacobians at the PCA and (2) add the time coordinate to the calculations.
Furthermore, we discuss an algorithm to calculate the 4D point on a track that exhibits the minimal 3D distance to a reference point (i.e., the 3D PCA). The purpose of this section is to elucidate the Athena code written by Giacinto Piacquadio while adding the time coordinate to his model. As usual, we consider both the absence of electromagnetic fields (where an exact analytical solution is possible) and the presence of a constant B-field in z-direction (where we use the Newton algorithm to find a solution iteratively).

2 General Notions

In the Perigee representation, a track is parametrized at its point of closest approach (PCA) P to the origin R of a reference coordinate system (see Fig. 3.1 and Fig. 4.1 for the definition of the points). The corresponding parameter vector $\mathbf{q}$ reads

$$\mathbf{q} := \begin{pmatrix} d_0 \\ z_0 \\ \varphi_P \\ \theta_P \\ (q/p)_P \\ t_P \end{pmatrix}, \tag{2.1}$$

where

- d_0 is the signed distance between P and R in the x-y plane
- $z_0 = z_P - z_R$ is the z-distance between P and R
- $\varphi_P \in [-\pi, \pi)$ is the polar angle of the momentum at P
- $\theta_P \in (0, \pi)$ is the azimuthal angle of the momentum at P
- $(q/p)_P$ is the charge of the particle divided by the absolute value of its momentum at P
- t_P is the track time at P

The sign convention for d_0 requires special care. We have

$$d_0 \begin{cases} > 0 \text{ if } \exists\, n \in \mathbb{Z} \text{ s.t. } \varphi_0 - \varphi_P = \frac{\pi}{2} + 2\pi n \\ < 0 \text{ otherwise} \end{cases},$$

where $\varphi_0 \in [-\pi, \pi)$ is the polar angle of the vector pointing from R to P. Note that for linear tracks (no EM fields) this translates to

$$\operatorname{sgn}(d_0) = \operatorname{sgn}(y_R - y_P), \tag{2.2}$$

and for helical tracks (constant B-field in z-direction, $\mathbf{B} = B\ \hat{\mathbf{e}}_z$) we have

$$\operatorname{sgn}(d_0) = \operatorname{sgn}(B) \operatorname{sgn}(q) \operatorname{sgn}\left(\rho^2 - (\mathbf{r}_R - \mathbf{r}_O)^2\right), \tag{2.3}$$

where ρ is the helix radius.[1]

One can write the six parameters from Eq. 2.1 as a function of the constant 4D reference point R, a 4D point on the track (point V in Fig. 3.1), and the momentum at the latter, e.g.:

$$d_0 = d_0(x_R, y_R, z_R, t_R, x_V, y_V, z_V, t_V, \varphi_V, \theta_V, (q/p)_V).$$

In the following, we will compute the Jacobian of the Perigee parameters in this representation, i.e.:

$$J := \left.\begin{pmatrix} \overbrace{\partial_{x_V} d_0 & \partial_{y_V} d_0 & \partial_{z_V} d_0 & \partial_{t_V} d_0}^{=:\,A} & \overbrace{\partial_{\varphi_V} d_0 & \partial_{\theta_V} d_0 & \partial_{(q/p)_V} d_0}^{=:\,B} \\ \partial_{x_V} z_0 & \ddots & & & & & \vdots \\ \partial_{x_V} \varphi & & & & & & \vdots \\ \partial_{x_V} \theta & & & & & & \vdots \\ \partial_{x_V} q/p & & & & & \ddots & \vdots \\ \partial_{x_V} t_P & \dots & \dots & \dots & \dots & \dots & \partial_{(q/p)_V} t_P \end{pmatrix}\right|_{V=P} \tag{2.4}$$

where we evaluate the Jacobian at the PCA P. We follow the literature convention and split the Jacobian into the submatrices A and B, which we call position and momentum Jacobian. Note that it is often useful to rewrite the derivative with respect to q/p like

$$\begin{aligned} \partial_{q/p} &= \partial_{q/p}\ p\ \partial_p \\ &= q\ \partial_{q/p} \left(\frac{q}{p}\right)^{-1} \partial_p \\ &= -q \left(\frac{q}{p}\right)^{-2} \partial_p \\ &= -\frac{p^2}{q} \partial_p, \end{aligned} \tag{2.5}$$

where we dropped the subscript for readability.

It is important to keep in mind that the Jacobian should only depend on the track parameters at the PCA. However, as we will see in the following, the terms involving time will often depend on the particle speed v, which cannot be extracted directly from $\mathbf{q}$. To obtain v nonetheless, we need to exploit a mass and a charge hypothesis since

$$v = c\beta = \frac{p}{\sqrt{p^2 + (cm_0)^2}},$$

where c is the speed of light and m_0 is the rest mass of the particle. While the mass hypothesis is needed to fix the value of m_0, the charge hypothesis allows us to extract the momentum p from the track parameters.

[1] $\mathrm{sgn}\left(\rho^2 - (\mathbf{r}_R - \mathbf{r}_O)^2\right)$ is (negative) positive if R is (outside) inside of the helix.

3 Absence of EM Fields

3.1 Track Linearization

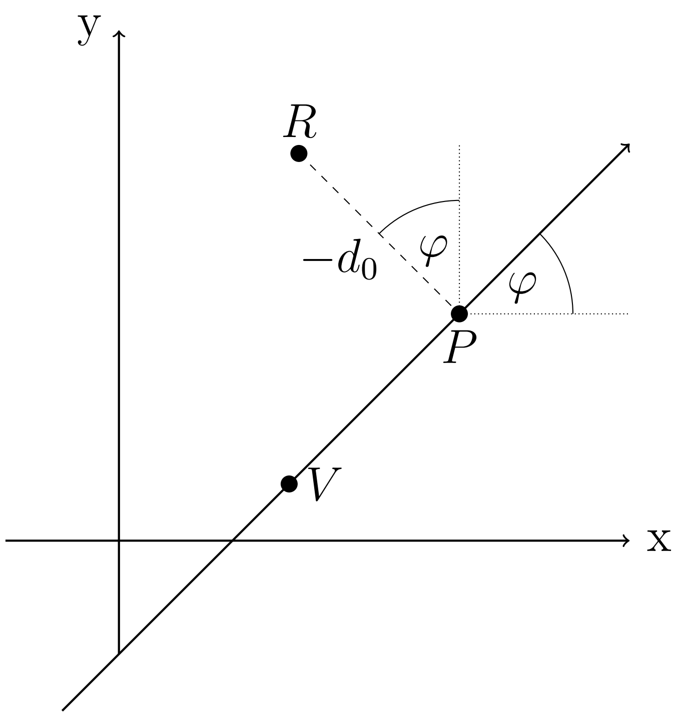

Figure 3.1: Projection of a track on the x-y plane in the absence of a magnetic field. The Perigee parametrization is given with respect to a coordinate system with origin in point R, whose axes are parallel to the global coordinate axes. d_0 is the x-y-distance between the reference point R and the PCA P of the trajectory to it. V denotes a general point on the trajectory. Note that we have $d_0 < 0$, $\varphi > 0$, and $\varphi_0 < 0$ in this plot.

If no electromagnetic field is present, the particle is not accelerated ($\ddot{\mathbf{r}} = 0$) and it thus moves on a straight trajectory, see Fig. 3.1. Therefore, φ, θ, and q/p are constant along the track, and we have

$$\varphi_V = \varphi_P =: \varphi$$
$$\theta_V = \theta_P =: \theta$$
$$(q/p)_V = (q/p)_P =: q/p$$

in the following.

Note that we perform all calculations for the situation shown in Fig. 3.1. One can (and should!) verify that we obtain the same results for different parameter signs and reference positions (e.g., when the particle moving in the opposite direction or when the reference R is below the track).

Let us start by expressing the coordinates of the PCA to the reference point R (i.e., the point P) with respect to the coordinates of the point V:

$$\mathbf{r}_P = \mathbf{r}_V + v \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} (t_P - t_V),$$

where v denotes the speed of the particle. Using the definition from Eq. 2.2 and keeping the sign of φ in mind, we can find another equation for $\mathbf{r}_P$:

$$\mathbf{r}_P = \mathbf{r}_R + \begin{pmatrix} d_0\sin\varphi \\ -d_0\cos\varphi \\ z_0 \end{pmatrix},$$

as one can easily verify from Fig. 3.1. Equating the two expressions for $\mathbf{r}_P$, we obtain:

$$\mathbf{r}_V + v \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} (t_P - t_V) = \mathbf{r}_R + \begin{pmatrix} d_0\sin\varphi \\ -d_0\cos\varphi \\ z_0 \end{pmatrix}. \tag{3.1}$$

Note that, the equation above contains only the Perigee parameters and the space-time coordinates of V.[1]

3.1.1 Derivatives of t_p

Before calculating the Jacobian, we must derive explicit functions for the Perigee parameters from Eq. 3.1. To obtain an expression for the time coordinate t, we rearrange the equations in the first two dimensions of Eq. 3.1:

$$\begin{aligned} d_0\sin\varphi &= x_V - x_R + v\sin\theta\cos\varphi\Delta t \\ -d_0\cos\varphi &= y_V - y_R + v\sin\theta\sin\varphi\Delta t, \end{aligned}$$

where we introduced $\Delta t \coloneqq t_P - t_V$. Division of the above equations furnishes:

$$\begin{aligned} -\tan\varphi &= \frac{x_V - x_R + v\sin\theta\cos\varphi\Delta t}{y_V - y_R + v\sin\theta\sin\varphi\Delta t} \\ -\tan\varphi\left(y_V - y_R + v\sin\theta\sin\varphi\Delta t\right) &= x_V - x_R + v\sin\theta\cos\varphi\Delta t \\ -\sin\varphi\left(y_V - y_R\right) - v\sin\theta\sin^2\varphi\Delta t &= \cos\varphi\left(x_V - x_R\right) + v\sin\theta\cos^2\varphi\Delta t, \end{aligned}$$

[1] The momentum at V coincides with the Perigee momentum due to the absence of a magnetic field.

where we multiplied by $\cos\varphi$ and used $\tan\varphi = \frac{\sin\varphi}{\cos\varphi}$ in the last step. We can simplify this expression by recalling that $\sin^2\varphi + \cos^2\varphi = 1$:

$$v\sin\theta\Delta t = -\cos\varphi\,(x_V - x_R) - \sin\varphi\,(y_V - y_R)\,.$$

Finally:

$$\begin{aligned} \Delta t &= -\frac{1}{v\sin\theta}\left(\cos\varphi\,(x_V - x_R) + \sin\varphi\,(y_V - y_R)\right) \\ \implies t_P &= t_V - \frac{1}{v\sin\theta}\left(\cos\varphi\,(x_V - x_R) + \sin\varphi\,(y_V - y_R)\right). \end{aligned} \tag{3.2}$$

We can now calculate the last row of the Jacobian from Eq. 2.4 using Eq. 3.2:

$$\begin{aligned} \partial_{x_V} t_P\Big|_{V=P} &= -\frac{\cos\varphi}{v\sin\theta} \\ \partial_{y_V} t_P\Big|_{V=P} &= -\frac{\sin\varphi}{v\sin\theta} \\ \partial_{z_V} t_P\Big|_{V=P} &= 0 \\ \partial_{t_V} t_P\Big|_{V=P} &= 1 \\ \partial_{\varphi} t_P\Big|_{V=P} &= \frac{1}{v\sin\theta}\left(\sin\varphi\,(x_V - x_R) - \cos\varphi\,(y_V - y_R)\right)\Big|_{V=P} \\ &= -\frac{d_0}{v\sin\theta} \\ \partial_{\theta} t_P\Big|_{V=P} &= -\left(\partial_\theta \frac{1}{v\sin\theta}\right)\left(\cos\varphi\,(x_V - x_R) + \sin\varphi\,(y_V - y_R)\right)\Big|_{V=P} \\ &= 0 \\ \partial_{q/p} t_P\Big|_{V=P} &= -\left(\partial_{q/p} \frac{1}{v\sin\theta}\right)\left(\cos\varphi\,(x_V - x_R) + \sin\varphi\,(y_V - y_R)\right)\Big|_{V=P} \\ &= 0, \end{aligned} \tag{3.3}$$

where we used that

$$\begin{aligned} (x_V - x_R)\Big|_{V=P} &= (x_P - x_R) \\ &= -\sin\varphi\ d_0 \\ (y_V - y_R)\Big|_{V=P} &= (y_P - y_R) \\ &= \quad \cos\varphi\ d_0. \end{aligned} \tag{3.4}$$

3.1.2 Derivatives of q/p

The fifth row of the Jacobian is obtained by noting that q/p is constant in the absence of an electric field and thus

$$\partial_{(q/p)_V} (q/p)_P\Big|_{V=P} = \partial_{q/p}\ q/p\Big|_{V=P} = 1,$$

while all other derivatives vanish.

3.1.3 Derivatives of θ

Again, θ is constant along the track in the absence of an electric field, and we have

$$\partial_{(\theta)_V}(\theta)_P\Big|_{V=P} = \partial_\theta\ \theta\Big|_{V=P} = 1,$$

while all other derivatives in the fourth row of the Jacobian vanish.

3.1.4 Derivatives of φ

φ is constant along the track in the absence of electric *and* magnetic field. Therefore, we find as before:

$$\partial_{(\varphi)_V}(\varphi)_P\Big|_{V=P} = \partial_\varphi\ \varphi\Big|_{V=P} = 1,$$

while all other derivatives in the third row of the Jacobian vanish.

3.1.5 Derivatives of z_0

To obtain an expression for z_0, we consider the third dimension of Eq. 3.1, i.e.:

$$z_V + v\cos\theta\Delta t = z_R + z_0.$$

Then,

$$z_0 = z_V - z_R + v\cos\theta\Delta t,$$

and we can find the derivatives of z_0 by exploiting

$$\partial_{q_i}\Delta t = \partial_{q_i} t_P - \delta_{q_i t_V},$$

in combination with the derivatives from Eq. 3.3. We find:

$$\begin{aligned}
\partial_{x_V} z_0\Big|_{V=P} &= v\cos\theta\ \partial_{x_V}\Delta t\Big|_{V=P} \\
&= -v\cos\theta\frac{\cos\varphi}{v\sin\theta} \\
&= -\cot\theta\cos\varphi \\
\partial_{y_V} z_0\Big|_{V=P} &= v\cos\theta\ \partial_{y_V}\Delta t\Big|_{V=P} \\
&= -v\cos\theta\frac{\sin\varphi}{v\sin\theta} \\
&= -\cot\theta\sin\varphi \\
\partial_{z_V} z_0\Big|_{V=P} &= 1 + v\cos\theta\ \partial_{z_V}\Delta t\Big|_{V=P} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\partial_{t_V} z_0\Big|_{V=P} &= v\cos\theta\ \partial_{t_V}\Delta t \\
&= 0 \\
\partial_{\varphi} z_0\Big|_{V=P} &= v\cos\theta\ \partial_{\varphi}\Delta t\Big|_{V=P} \\
&= -v\cos\theta \frac{d_0}{v\sin\theta} \\
&= -d_0\cot\theta \\
\partial_{\theta} z_0\Big|_{V=P} &= -v\sin\theta\Delta t\Big|_{V=P} + v\cos\theta\ \partial_{\theta}\Delta t\Big|_{V=P} \\
&= 0 \\
\partial_{q/p} z_0\Big|_{V=P} &= v\cos\theta\ \partial_{q/p}\Delta t\Big|_{V=P} \\
&= 0,
\end{aligned}$$

where we used that

$$\Delta t\Big|_{V=P} = (t_P - t_V)\Big|_{V=P} = 0. \tag{3.5}$$

3.1.6 Derivatives of d_0

An expression for d_0 can be found by rearranging the first two dimensions of Eq. 3.1 like:

$$\begin{aligned}
d_0\sin\varphi - v\sin\theta\cos\varphi\Delta t &= x_V - x_R \\
-d_0\cos\varphi - v\sin\theta\sin\varphi\Delta t &= y_V - y_R.
\end{aligned}$$

Squaring and adding these equations furnishes

$$d_0^2 + v^2\sin^2\theta(\Delta t)^2 = (x_V - x_R)^2 + (y_V - y_R)^2,$$

which could have been deduced geometrically by noting that the speed in the x-y-plane is given by $v_T = v\sin\theta$ and by applying the Pythagorean theorem in Fig. 3.1. Solving for d_0 furnishes

$$|d_0| = \sqrt{(x_V - x_R)^2 + (y_V - y_R)^2 - v^2\sin^2\theta(\Delta t)^2},$$

and, by using Eq. 2.2,

$$d_0 = \operatorname{sgn}(y_R - y_P)\sqrt{(x_V - x_R)^2 + (y_V - y_R)^2 - v^2\sin^2\theta(\Delta t)^2}.$$

The derivatives of d_0 read

$$
\begin{aligned}
\partial_{x_V} d_0\Big|_{V=P} &= \frac{1}{d_0}(x_V - x_R - v^2 \sin^2\theta \Delta t\ \partial_{x_V}\Delta t)\Big|_{V=P} \\
&= \frac{x_P - x_R}{d_0} \\
&= \frac{-\sin\varphi d_0}{d_0} \\
&= -\sin\varphi \\
\partial_{y_V} d_0\Big|_{V=P} &= \frac{1}{d_0}(y_V - y_R - v^2 \sin^2\theta \Delta t\ \partial_{y_V}\Delta t)\Big|_{V=P} \\
&= \frac{y_P - y_R}{d_0} \\
&= \frac{\cos\varphi d_0}{d_0} \\
&= \cos\varphi \\
\partial_{z_V} d_0\Big|_{V=P} &= \frac{1}{d_0}(-v^2 \sin^2\theta \Delta t\ \partial_{z_V}\Delta t)\Big|_{V=P} \\
&= 0 \\
\partial_{t_V} d_0\Big|_{V=P} &= \frac{1}{d_0}(-v^2 \sin^2\theta \Delta t\ \partial_{t_V}\Delta t)\Big|_{V=P} \\
&= 0 \\
\partial_{\varphi} d_0\Big|_{V=P} &= \frac{1}{d_0}(-v^2 \sin^2\theta \Delta t\ \partial_{\varphi}\Delta t)\Big|_{V=P} \\
&= 0 \\
\partial_{\theta} d_0\Big|_{V=P} &= \frac{1}{d_0}(-v^2 \sin^2\theta \Delta t\ \partial_{\theta}\Delta t)\Big|_{V=P} \\
&= 0 \\
\partial_{q/p} d_0\Big|_{V=P} &= \frac{1}{d_0}(-v^2 \sin^2\theta \Delta t\ \partial_{q/p}\Delta t)\Big|_{V=P} \\
&= 0,
\end{aligned}
$$

where we used Eqs. 3.4 and 3.5.

3.1.7 Results

Summing up the results from the previous sections, the position Jacobian reads:

$$A = \begin{pmatrix} -\sin\varphi & \cos\varphi & 0 & 0 \\ -\cot\theta\cos\varphi & -\cot\theta\sin\varphi & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\cos\varphi}{v_T} & -\frac{\sin\varphi}{v_T} & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ -d_0\cot\theta & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{d_0}{v_T} & 0 & 0 \end{pmatrix}, \tag{3.6}$$

where $v_T \equiv v\sin\theta$ is the speed in the x-y-plane. When comparing to Eq 5.40 from Ref. [1], we note that several terms in the momentum Jacobian differ from our results. This is because we evaluate the Jacobian at the PCA P while Ref. [1] evaluates the Jacobian at a general point on the trajectory V.[2]

3.1.8 Special Case $P = R$

In the case where the track passes through the reference point, i.e., $P = R$, the momentum Jacobian can be simplified further since $d_0 = 0$:

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.7}$$

3.2 3D PCA

Given a fixed point V on the track, we can calculate its 3D PCA P' to a reference position R' analytically. $\mathbf{r}_{P'}$ can be obtained as a function of the time difference $\Delta t := t_{P'} - t_V$ and the constant velocity of the particle:

$$\mathbf{r}_{P'} = \mathbf{r}_V + \mathbf{v}\Delta t. \tag{3.8}$$

The distance between P' and R' is minimal if

$$(\mathbf{r}_{R'} - \mathbf{r}_{P'}) \cdot \hat{\mathbf{e}}_{\mathbf{v}} = 0,$$

[2] Note that, for all practical applications, we perform the linearization at the PCA.

where $\hat{\mathbf{e}}_{\mathbf{v}}$ is a unit vector in the direction of the velocity. Using Eq. 3.8, we have

$$(\mathbf{r}_{R'} - \mathbf{r}_V - \mathbf{v}\Delta t) \cdot \hat{\mathbf{e}}_{\mathbf{v}} = 0,$$

and thus

$$t_{P'} = t_V + \frac{1}{v}(\mathbf{r}_{R'} - \mathbf{r}_V) \cdot \hat{\mathbf{e}}_{\mathbf{v}}, \tag{3.9}$$

where $v = |\mathbf{v}|$. Plugging Eq. 3.9 back into Eq. 3.8, we obtain

$$\mathbf{r}_{P'} = \mathbf{r}_V + \left((\mathbf{r}_{R'} - \mathbf{r}_V) \cdot \hat{\mathbf{e}}_{\mathbf{v}}\right)\ \hat{\mathbf{e}}_{\mathbf{v}}.$$

4 Constant Magnetic Field

4.1 Track Linearization

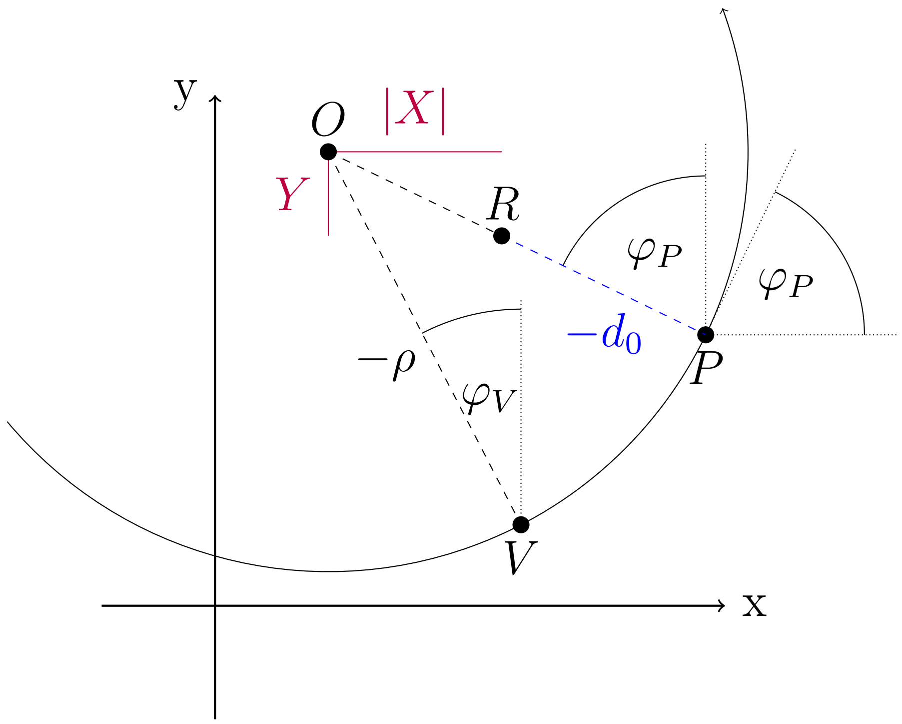

Figure 4.1: Projection of a track on the x-y plane in a constant magnetic field in z-direction. The particle moves counterclockwise on a helix with radius $|\rho|$ (i.e., a negative (positive) particle is moving in a B-field in positive (negative) z-direction). The Perigee parametrization is given with respect to a coordinate system with origin in point R, whose axes are parallel to the global coordinate axes. d_0 is the x-y-distance between the reference point R and the PCA P of the trajectory to it. V denotes a general point on the trajectory. Note that we have $d_0 < 0$, $\rho < 0$, $\varphi_P > 0$, $\varphi_V > 0$, $X < 0$, and $Y > 0$ in this plot. For the angle φ_0 between the x-axis and the vector from R to P, we have $\varphi_0 < 0$.

For a constant B-field in z-direction, the differential equations governing the particle movement read

$$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} = \begin{pmatrix} \dot{y}B \\ -\dot{x}B \\ 0 \end{pmatrix}. \tag{4.1}$$

Note that the acceleration in the transverse plane (i.e., the x-y-plane) is always perpendicular to the velocity in said plane. Therefore, the speed in the transverse plane $v_T \equiv \sqrt{\dot{x}^2 + \dot{y}^2}$ is constant. Furthermore, there is no acceleration in z-direction and thus the speed in said direction $v_z \equiv \dot{z}$ is constant. Consequently, the total speed $v = \sqrt{v_T^2 + v_z^2}$ is also constant. This allows us to conclude that $\theta = \arcsin(v_T/v)$ and $q/p = q/(mv)$ are constant as well, and we can write

$$\begin{aligned} \theta_V = \theta_P &=: \theta \\ (q/p)_V = (q/p)_P &=: q/p \end{aligned}$$

in the following calculations.

Choosing the initial conditions

$$\begin{aligned} x(0) &= x_{t=0} \quad & y(0) &= y_{t=0} \quad & z(0) &= z_{t=0} \\ \dot{x}(0) &= v \sin\theta \quad & \dot{y}(0) &= 0 \quad & \dot{z}(0) &= v\cos\theta, \end{aligned}$$

we find

$$\begin{aligned} x(t) &= x_{t=0} + \rho \sin(\omega_0 t) \\ y(t) &= y_{t=0} + \rho (\cos(\omega_0 t) - 1) \\ z(t) &= z_{t=0} + v \cos\theta \; t \end{aligned} \tag{4.2}$$

as solution for Eq. 4.1. The particle thus follows a helix with radius

$$\begin{aligned} \rho &= \frac{mv\sin\theta}{qB} \\ &= \frac{p\sin\theta}{qB} \\ &= \frac{p_T}{qB} \end{aligned} \tag{4.3}$$

and angular frequency

$$\omega_0 = \frac{qB}{m}.$$

Note that the sign of the radius depends on the direction of the B-field and on the charge of the particle. For example, if the B-field is oriented in positive z-direction, (counter)clockwise rotation corresponds to (negative) positive charge and consequently to (negative) positive ρ. Following the literature convention, we define:

$$h \coloneqq \operatorname{sgn}(\rho) = \operatorname{sgn}(q)\operatorname{sgn}(B)\,. \tag{4.4}$$

Comparing to Eq. 2.3, we then obtain

$$\operatorname{sgn}(d_0) = h\ \operatorname{sgn}\left(\rho^2 - (\mathbf{r}_R - \mathbf{r}_O)^2\right).$$

One can relate the radius and the angular momentum like

$$\frac{1}{\omega_0} = \frac{\rho}{v\sin\theta}, \tag{4.5}$$

which will become useful later on. The particle velocity can be retrieved by differentiating Eqs. 4.2:

$$\begin{aligned} \dot{x}(t) &= \quad \rho\omega_0\cos(\omega_0 t) \\ \dot{y}(t) &= -\rho\omega_0\sin(\omega_0 t) \\ \dot{z}(t) &= \quad v\cos\theta. \end{aligned} \tag{4.6}$$

Like in Sec. 3.1, we want to express the Perigee parameters as a function of the free parameters at V. Note that we perform all calculations for the situation shown in Fig. 4.1. One can (and should!) verify that we obtain the same results for different parameter signs and reference positions (e.g., when the particle moving clockwise or when the reference R is in a different quadrant or outside of the helix).

4.1.1 Derivatives of φ_P

We start by finding an expression for φ_P, which is a convenient choice as we will see a little further down the road. From Fig. 4.1 we find

$$\begin{aligned} x_V &= x_R + |d_0|\sin|\varphi_P| - |\rho|\sin|\varphi_P| + |\rho|\sin|\varphi_V| \\ y_V &= y_R - |d_0|\cos|\varphi_P| + |\rho|\cos|\varphi_P| - |\rho|\cos|\varphi_V|, \end{aligned}$$

and, using the correct signs for the parameters,

$$\begin{aligned} x_V &= x_R - d_0\sin\varphi_P + \rho\sin\varphi_P - \rho\sin\varphi_V \\ y_V &= y_R + d_0\cos\varphi_P - \rho\cos\varphi_P + \rho\cos\varphi_V. \end{aligned} \tag{4.7}$$

Rearranging furnishes

$$\begin{aligned} -\sin\varphi_P(d_0 - \rho) &= x_V - x_R + \rho\sin\varphi_V \\ \cos\varphi_P(d_0 - \rho) &= y_V - y_R - \rho\cos\varphi_V, \end{aligned}$$

and, by dividing the equations,

$$\begin{aligned} -\tan\varphi_P &= \frac{x_V - x_R + \rho\sin\varphi_V}{y_V - y_R - \rho\cos\varphi_V} \\ &\equiv \frac{X}{Y}, \end{aligned} \tag{4.8}$$

where we defined

$$\begin{aligned} X &\coloneqq x_V - x_R + \rho\sin\varphi_V \\ Y &\coloneqq y_V - y_R - \rho\cos\varphi_V. \end{aligned} \tag{4.9}$$

Using the relation

$$-\tan x = \frac{1}{\tan(x + \pi/2)},$$

we conclude

$$\varphi_P = \arctan\left(\frac{Y}{X}\right) - \frac{\pi}{2}. \tag{4.10}$$

Note that X and Y are the x- and y-coordinate of the helix center O in the reference coordinate system with origin in R.[1] Consequently, X and Y are independent of where we place the point V on the track, and we can write

$$\begin{aligned} X_V &\equiv X \\ Y_V &\equiv Y, \end{aligned}$$

as the choice of notation in Eq. 4.9 already hinted. It is convenient to define the distance S between O and R:

$$S \coloneqq \sqrt{X^2 + Y^2}.$$

We can then express X and Y via S:

$$\begin{aligned} X &= \quad hS\sin\varphi_P \\ Y &= -hS\cos\varphi_P, \end{aligned} \tag{4.11}$$

where h is the sign of the helix radius as defined in Eq. 4.4.

Let us compute some derivatives of these quantities. We have

$$\begin{aligned} \partial_\theta\rho &= \frac{mv\cos\theta}{qB} \\ &= \rho\cot\theta, \\ \partial_{q/p}\rho &= -\frac{p^2}{q}\partial_p\rho \\ &= -\frac{p^2}{q}\frac{\rho}{p} \\ &= -\frac{\rho}{q/p} \end{aligned}$$

[1] Applying this knowledge to Fig. 4.1 confirms Eq. 4.8 geometrically.

while all other derivatives of ρ vanish. Therefore, from Eq. 4.9,

$$\begin{aligned}
\partial_{x_V} X &= 1 \\
\partial_{\varphi_V} X &= \rho \cos \varphi_V \\
\partial_\theta X &= \rho \cot \theta \sin \varphi_V, \\
\partial_{q/p} X &= -\frac{\rho}{q/p} \sin \varphi_V,
\end{aligned}$$

and

$$\begin{aligned}
\partial_{y_V} Y &= 1 \\
\partial_{\varphi_V} Y &= \rho \sin \varphi_V \\
\partial_\theta Y &= -\rho \cot \theta \cos \varphi_V, \\
\partial_{q/p} Y &= \frac{\rho}{q/p} \cos \varphi_V
\end{aligned}$$

while all other derivatives of X and Y vanish. Keeping in mind that

$$\partial_x \arctan x = \frac{1}{1+x^2},$$

we can derive φ_P with respect to X and Y:

$$\begin{aligned}
\partial_X \varphi_P &= \frac{1}{1+\left(\frac{Y}{X}\right)^2} \left(-\frac{Y}{X^2}\right) \\
&= -\frac{Y}{S^2} \\
\partial_Y \varphi_P &= \frac{1}{1+\left(\frac{Y}{X}\right)^2} \frac{1}{X} \\
&= \frac{X}{S^2}
\end{aligned}$$

Finally, we put all pieces together to compute the third row of the Jacobian:

$$\begin{aligned}
\partial_{x_V} \varphi_P \Big|_{V=P} &= \partial_{x_V} X \partial_X \varphi_P \Big|_{V=P} \\
&= -\frac{Y}{S^2} \\
\partial_{y_V} \varphi_P \Big|_{V=P} &= \partial_{y_V} Y \partial_Y \varphi_P \Big|_{V=P} \\
&= \frac{X}{S^2} \\
\partial_{z_V} \varphi_P \Big|_{V=P} &= 0 \\
\partial_{t_V} \varphi_P \Big|_{V=P} &= 0
\end{aligned}$$

$$
\begin{aligned}
\partial_{\varphi_V}\varphi_P\Big|_{V=P} &= \left(\partial_{\varphi_V}X\partial_X\varphi_P + \partial_{\varphi_V}Y\partial_Y\varphi_P\right)\Big|_{V=P} \\
&= \rho\left(\frac{-Y\cos\varphi_V + X\sin\varphi_V}{S^2}\right)\Bigg|_{V=P} \\
&= \rho\left(\frac{h\cos\varphi_P\cos\varphi_V + h\sin\varphi_P\sin\varphi_V}{S}\right)\Bigg|_{V=P} \\
&= h\frac{\rho}{S} \\
&= \frac{|\rho|}{S} \\
\partial_\theta\varphi_P\Big|_{V=P} &= \left(\partial_\theta X\partial_X\varphi_P + \partial_\theta Y\partial_Y\varphi_P\right)\Big|_{V=P} \\
&= \rho\cot\theta\left(\frac{-Y\sin\varphi_V - X\cos\varphi_V}{S^2}\right)\Bigg|_{V=P} \\
&= \rho\cot\theta\left(\frac{h\cos\varphi_P\sin\varphi_V - h\sin\varphi_P\cos\varphi_V}{S}\right)\Bigg|_{V=P} \\
&= 0, \\
\partial_{q/p}\varphi_P\Big|_{V=P} &= \left(\partial_{q/p}X\partial_X\varphi_P + \partial_{q/p}Y\partial_Y\varphi_P\right)\Big|_{V=P} \\
&= \frac{\rho}{q/p}\left(\frac{Y\sin\varphi_V + X\cos\varphi_V}{S^2}\right)\Bigg|_{V=P} \\
&= \frac{\rho}{q/p}\left(\frac{-h\cos\varphi_P\sin\varphi_V + h\sin\varphi_P\cos\varphi_V}{S}\right)\Bigg|_{V=P} \\
&= 0,
\end{aligned}
$$

where we used the chain rule and Eq. 4.11.

4.1.2 Derivatives of t_P

Let us continue by computing the last row of the Jacobian. From Fig. 4.1 we find geometrically

$$\tan\varphi_V = \frac{\dot{y}(t_V)}{\dot{x}(t_V)}.$$

Using the expressions from Eq. 4.6 allows us to obtain a relation between the time and the polar angle φ:

$$
\begin{aligned}
&\tan\varphi_V = -\tan\left(\omega_0 t_V\right) \\
\implies &\varphi_V + 2\pi n_V = -\omega_0 t_V, \;\; n_V \in \mathbb{N}.
\end{aligned} \tag{4.12}
$$

Note that

$$n_V \to n_V + 1 \text{ iff } \varphi_V = -\pi,$$

and thus

$$\partial_{\varphi_V} n_V = \delta\left(\varphi_V + \pi\right).$$

Then

$$\begin{aligned}
\Delta t &\equiv t_P - t_V \\
&= -\frac{1}{\omega_0}(\varphi_P - \varphi_V + 2\pi(n_P - n_V)) \\
\implies t_P &= t_V - \frac{\rho}{v\sin\theta}(\varphi_P - \varphi_V + 2\pi(n_P - n_V)),
\end{aligned} \tag{4.13}$$

where we used Eq. 4.5 to replace the angular frequency by the helix radius. The derivatives of t_P follow directly from the calculations for φ_P from Sec. 4.1.1:

$$\begin{aligned}
\partial_{x_V} t_P\Big|_{V=P} &= -\frac{\rho}{v\sin\theta}\partial_{x_V}\varphi_P\Big|_{V=P} \\
&= \frac{\rho}{v\sin\theta}\frac{Y}{S^2} \\
\partial_{y_V} t_P\Big|_{V=P} &= -\frac{\rho}{v\sin\theta}\partial_{y_V}\varphi_P\Big|_{V=P} \\
&= -\frac{\rho}{v\sin\theta}\frac{X}{S^2} \\
\partial_{z_V} t_P\Big|_{V=P} &= -\frac{\rho}{v\sin\theta}\partial_{z_V}\varphi_P\Big|_{V=P} \\
&= 0 \\
\partial_{t_V} t_P\Big|_{V=P} &= 1 - \frac{\rho}{v\sin\theta}\partial_{t_V}\varphi_P\Big|_{V=P} \\
&= 1 \\
\partial_{\varphi_V} t_P\Big|_{V=P} &= -\frac{\rho}{v\sin\theta}\left(\partial_{\varphi_V}\varphi_P - 1 + 2\pi(\partial_{\varphi_V}\varphi_P\ \delta\left(\varphi_P + \pi\right) - \delta\left(\varphi_V + \pi\right))\right)\Big|_{V=P} \\
&= \frac{\rho}{v\sin\theta}\left(1 - \frac{|\rho|}{S}\right)\left(1 + 2\pi\delta\left(\varphi_P + \pi\right)\right) \\
\partial_{\theta} t_P\Big|_{V=P} &= -\left(\partial_\theta\frac{\rho}{v\sin\theta}\right)\left(\varphi_P - \varphi_V + 2\pi(n_P - n_V)\right)\Big|_{V=P} \\
&= 0 \\
\partial_{q/p} t_P\Big|_{V=P} &= -\frac{\rho}{v\sin\theta}\partial_{q/p}\varphi_P\Big|_{V=P} \\
&= 0,
\end{aligned}$$

where we used that

$$\begin{aligned}
\left(\varphi_P - \varphi_V + 2\pi(n_P - n_V)\right)\Big|_{V=P} &= \left(\varphi_P - \varphi_P + 2\pi(n_P - n_P)\right) \\
&= 0.
\end{aligned}$$

4.1.3 Derivatives of q/p

As in Sec. 3.1, the fifth row of the Jacobian is obtained by noting that q/p is constant in the absence of an electric field and thus

$$\partial_{(q/p)_V}(q/p)_P\Big|_{V=P} = \partial_{q/p}\ q/p\Big|_{V=P} = 1,$$

while all other derivatives vanish.

4.1.4 Derivatives of θ

θ is constant along the track in the absence of an electric field, and we have

$$\partial_{(\theta)_V}(\theta)_P\Big|_{V=P} = \partial_\theta\ \theta\Big|_{V=P} = 1,$$

while all other derivatives in the fourth row of the Jacobian vanish.

4.1.5 Derivatives of z_0

From the third equation of Eq. 4.2, we have:

$$\begin{aligned} z_V &= z_P - v\cos\theta(t_P - t_V) \\ &= z_R + z_0 - v\cos\theta(t_P - t_V) \\ \implies z_0 &= z_V - z_R - \rho\cot\theta\ (\varphi_P - \varphi_V + 2\pi(n_P - n_V)) \end{aligned} \tag{4.14}$$

where we plugged in the definition of z_0 in the second step and used Eq. 4.13 in the third step. The derivatives of z_0 are then obtained from the derivatives of φ_P from Sec. 4.1.1:

$$\begin{aligned} \partial_{x_V} z_0\Big|_{V=P} &= -\rho\cot\theta\ \partial_{x_V}\varphi_P\Big|_{V=P} \\ &= -\rho\cot\theta\left(-\frac{Y}{S^2}\right) \\ &= \rho\cot\theta\frac{Y}{S^2} \\ \partial_{y_V} z_0\Big|_{V=P} &= -\rho\cot\theta\ \partial_{y_V}\varphi_P\Big|_{V=P} \\ &= -\rho\cot\theta\frac{X}{S^2} \\ \partial_{z_V} z_0\Big|_{V=P} &= 1 \\ \partial_{t_V} z_0\Big|_{V=P} &= 0 \\ \partial_{\varphi_V} z_0\Big|_{V=P} &= -\rho\cot\theta\left(\partial_{\varphi_V}\varphi_P - 1 + 2\pi(\partial_{\varphi_V}\varphi_P\ \delta(\varphi_P + \pi) - \delta(\varphi_V + \pi))\right)\Big|_{V=P} \\ &= \rho\cot\theta\left(1 - \frac{|\rho|}{S}\right)(1 + 2\pi\delta(\varphi_P + \pi)) \end{aligned}$$

$$\begin{aligned}
\partial_\theta z_0\Big|_{V=P} &= -\left(\partial_\theta(\rho\cot\theta)\right)\left(\varphi_P - \varphi_V + 2\pi(n_P - n_V)\right)\Big|_{V=P} \\
&= 0 \\
\partial_{q/p} z_0\Big|_{V=P} &= -\left(\partial_{q/p}\rho\right)\cot\theta(\varphi_P - \varphi_V + 2\pi(n_P - n_V))\Big|_{V=P} \\
&= 0.
\end{aligned}$$

4.1.6 Derivatives of d_0

To find an expression for d_0, we can rearrange Eqs. 4.7 like

$$\begin{aligned}
\sin\varphi_P(\rho - d_0) &= x_V - x_R + \rho\sin\varphi_V \\
&\equiv X \\
-\cos\varphi_P(\rho - d_0) &= y_V - y_R - \rho\cos\varphi_V \\
&\equiv Y.
\end{aligned}$$

Squaring and adding the two equations leads to

$$\begin{aligned}
(\rho - d_0)^2 &= X^2 + Y^2, \\
&\equiv S^2,
\end{aligned}$$

which is what one would expect from geometrical considerations. Taking the square root furnishes

$$\begin{aligned}
d_0 &= \rho - \mathrm{sgn}\,(\rho - d_0)\,S \\
&= \rho - \mathrm{sgn}\,(\rho)\,S \\
&\equiv \rho - hS
\end{aligned} \tag{4.15}$$

Let us proof the second equality.

Proof. *We need to consider four cases:*

- *R is in the helix center ($R = O$)*
  $\Longrightarrow$ *$S = 0$ and the equality holds.*

- *R is inside the helix but not in the helix center*
  $\Longrightarrow\ sgn\,(\rho) = sgn\,(d_0)\,,\ |\rho| > |d_0|$
  $\Longrightarrow\ sgn\,(\rho - d_0) = sgn\,(sgn\,(\rho)\,(|\rho| - |d_0|)) = sgn\,(\rho)$

- *R is on the helix*
  $\Longrightarrow\ d_0 = 0$
  $\Longrightarrow\ sgn\,(\rho - d_0) = sgn\,(\rho)$

- *R is outside the helix*
  $\Longrightarrow\ sgn\,(\rho) = -sgn\,(d_0)$
  $\Longrightarrow\ sgn\,(\rho - d_0) = sgn\,(sgn\,(\rho)\,(|\rho| + |d_0|)) = sgn\,(\rho)$

To compute the derivatives of Eq. 4.15, it is useful to note that

$$\partial_X S = \frac{X}{S}$$
$$\partial_Y S = \frac{Y}{S}.$$

Furthermore, thanks to Eq. 4.4, all other derivatives of h vanish.

Then, by virtue of the chain rule,

$$\begin{aligned}
\partial_{x_v} d_0\Big|_{V=P} &= \partial_{x_V} X \partial_X d_0\Big|_{V=P} \\
&= -h\ \partial_X S\Big|_{V=P} \\
&= -h\frac{X}{S} \\
\partial_{y_v} d_0\Big|_{V=P} &= \partial_{y_V} Y \partial_Y d_0\Big|_{V=P} \\
&= -h\ \partial_Y S\Big|_{V=P} \\
&= -h\frac{Y}{S} \\
\partial_{z_v} d_0\Big|_{V=P} &= 0 \\
\partial_{t_V} d_0\Big|_{V=P} &= 0 \\
\partial_{\varphi_V} d_0\Big|_{V=P} &= \left(\partial_{\varphi_V} X \partial_X d_0 + \partial_{\varphi_V} Y \partial_Y d_0\right)\Big|_{V=P} \\
&= \left(\rho\cos\varphi_V\left(-h\frac{X}{S}\right) + \rho\sin\varphi_V\left(-h\frac{Y}{S}\right)\right)\Bigg|_{V=P} \\
&= -h\rho\left(\frac{X\cos\varphi_V + Y\sin\varphi_V}{S}\right)\Bigg|_{V=P} \\
&= -h\rho\left(h\sin\varphi_P\cos\varphi_V - h\cos\varphi_P\sin\varphi_V\right)\Big|_{V=P} \\
&= 0 \\
\partial_\theta d_0\Big|_{V=P} &= \left(\partial_\theta\rho - h\left(\partial_\theta X\partial_X S + \partial_\theta Y\partial_Y S\right)\right)\Big|_{V=P} \\
&= \rho\cot\theta - h\rho\cot\theta\left(\sin\varphi_V\frac{X}{S} - \cos\varphi_V\frac{Y}{S}\right)\Bigg|_{V=P} \\
&= \rho\cot\theta - h\rho\cot\theta\left(\sin\varphi_V(h\sin\varphi_P) - \cos\varphi_V(-h\cos\varphi_P)\right)\Big|_{V=P} \\
&= \rho\cot\theta - h^2\rho\cot\theta \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\partial_{q/p} d_0 \Big|_{V=P} &= \left(\partial_{q/p}\rho - h\left(\partial_{q/p}X\partial_X S + \partial_{q/p}Y\partial_Y S\right)\right)\Big|_{V=P} \\
&= -\frac{\rho}{q/p} - h\frac{\rho}{q/p}\left(-\sin\varphi_V \frac{X}{S} + \cos\varphi_V \frac{Y}{S}\right)\Big|_{V=P} \\
&= -\frac{\rho}{q/p} - h\frac{\rho}{q/p}\left(-\sin\varphi_V(h\sin\varphi_P) + \cos\varphi_V(-h\cos\varphi_P)\right)\Big|_{V=P} \\
&= -\frac{\rho}{q/p} + h^2\frac{\rho}{q/p} \\
&= 0,
\end{aligned}
$$

where we used Eq. 4.11.

4.1.7 Results

Neglecting the terms containing Kronecker deltas, the position and the momentum Jacobian for helical tracks read

$$
\begin{aligned}
A &= \begin{pmatrix}
-h\frac{X}{S} & -h\frac{Y}{S} & 0 & 0 \\
\rho\cot\theta\frac{Y}{S^2} & -\rho\cot\theta\frac{X}{S^2} & 1 & 0 \\
-\frac{Y}{S^2} & \frac{X}{S^2} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{\rho}{v_T}\frac{Y}{S^2} & -\frac{\rho}{v_T}\frac{X}{S^2} & 0 & 1
\end{pmatrix}, \\
B &= \begin{pmatrix}
0 & 0 & 0 \\
\rho\cot\theta\left(1-\frac{|\rho|}{S}\right) & 0 & 0 \\
\frac{|\rho|}{S} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\frac{\rho}{v_T}\left(1-\frac{|\rho|}{S}\right) & 0 & 0
\end{pmatrix},
\end{aligned}
\tag{4.16}
$$

where $v_T \equiv v\sin\theta$ is the speed in the x-y-plane. When comparing to Eq 5.36 from Ref. [1], we note that several terms in the momentum Jacobian differ from our results. This is because we evaluate the Jacobian at the PCA P while Ref. [1] evaluates the Jacobian at a general point on the trajectory V.[2]

[2]Note that, for all practical applications, we perform the linearization at the PCA.

Note that the terms in the first row can be simplified even further. We have

$$\begin{aligned} -h\frac{X}{S} &= -h\frac{hS\sin\varphi_P}{S} \\ &= -\sin\varphi_P \\ -h\frac{Y}{S} &= -h\frac{-hS\cos\varphi_P}{S} \\ &= \cos\varphi_P, \end{aligned} \tag{4.17}$$

where we used Eq. 4.11.

4.1.8 Asymptotic Behavior

Let us briefly discuss what happens when $|\rho| \to \infty$, i.e., when $|B|$ is small compared to $|p_T/q|$ (Eq. 4.3). From Fig. 4.1, we see that

$$S \xrightarrow{|\rho|\to\infty} \infty.$$

Rearranging Eq. 4.15, we obtain

$$\frac{|\rho|}{S} = 1 + \frac{d_0 h}{S} \xrightarrow{|\rho|\to\infty} 1,$$

since d_0 is constant for $|\rho| \to \infty$. Then, we can write

$$\begin{aligned} \rho\frac{X}{S^2} &= \rho\frac{hS\sin\varphi_P}{S^2} \\ &= |\rho|\frac{\sin\varphi_P}{S} \\ &\xrightarrow{|\rho|\to\infty} \sin\varphi_P \\ \rho\frac{Y}{S^2} &= \rho\frac{-hS\cos\varphi_P}{S^2} \\ &= -|\rho|\frac{\cos\varphi_P}{S} \\ &\xrightarrow{|\rho|\to\infty} -\cos\varphi_P, \end{aligned} \tag{4.18}$$

where we used Eq. 4.11. Consequently,

$$\begin{aligned} \frac{X}{S^2} &\xrightarrow{|\rho|\to\infty} 0 \\ \frac{Y}{S^2} &\xrightarrow{|\rho|\to\infty} 0. \end{aligned}$$

Furthermore,

$$\begin{aligned}\rho\left(1-\frac{|\rho|}{S}\right) &= h\rho\left(h-\frac{\rho}{S}\right)\\ &= \frac{h\rho}{S}\left(hS-\rho\right)\\ &= \frac{|\rho|}{S}(-d_0)\\ &\xrightarrow{\rho\to\infty} -d_0,\end{aligned}$$

where we used Eq. 4.15. Plugging these results into Eq. 4.16 and using Eq. 4.17, we retrieve the Jacobians in the absence of a magnetic field from Eq. 3.6.

4.1.9 Special Case $P = R$

In the case where the track passes exactly through the reference point, i.e., $P = R$, the Jacobians from Eq. 4.16 can be simplified. Imagining that R sits on top of P in Fig. 4.1, we find geometrically:

$$S = \sqrt{X^2 + Y^2} = |\rho|.$$

Therefore, we can redo the same calculation as in Eq. 4.18 to obtain

$$\begin{aligned}\rho\frac{X}{S^2} &= \sin\varphi_P\\ \rho\frac{Y}{S^2} &= -\cos\varphi_P.\end{aligned}$$

Note that we did not have to take the limit this time and the result also holds for finite ρ. Using this result, the Jacobians can be rewritten

$$A = \begin{pmatrix} -\sin\varphi_P & \cos\varphi_P & 0 & 0\\ -\cot\theta\cos\varphi_P & -\cot\theta\sin\varphi_P & 1 & 0\\ \frac{\cos\varphi_P}{\rho} & \frac{\sin\varphi_P}{\rho} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\frac{\cos\varphi_P}{v_T} & -\frac{\sin\varphi_P}{v_T} & 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{pmatrix}.$$

For $\rho \to \infty$, this result simplifies to the case where $B = 0$, see Eqs. 3.6 and 3.7.

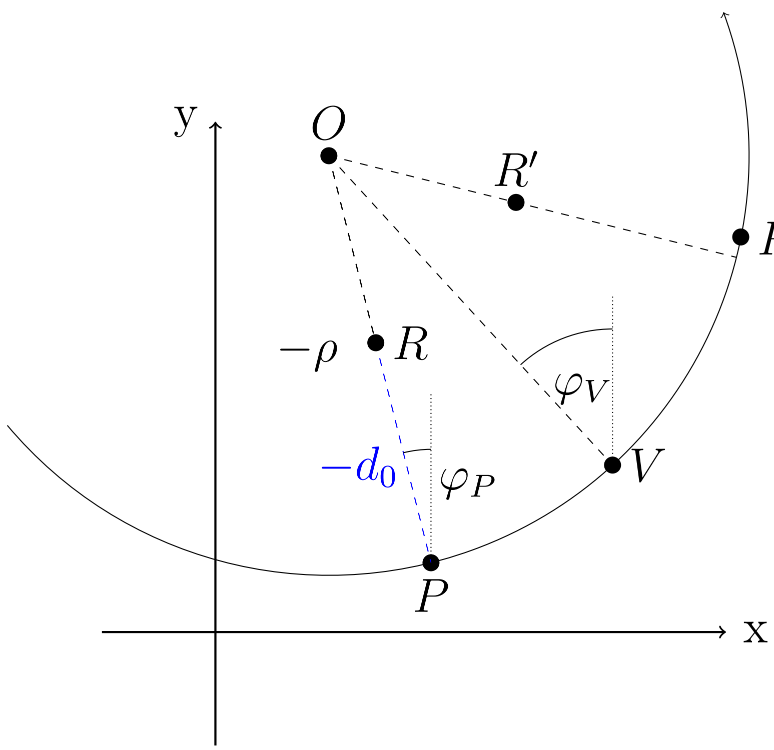

Figure 4.2: Projection of a track on the x-y plane in a constant magnetic field in z-direction. The particle moves counterclockwise on a helix with radius $|\rho|$ (i.e., a negative (positive) particle is moving in a B-field in positive (negative) z-direction). We discuss an algorithm that starts at the 2D PCA P of a reference point R and converges towards the 3D PCA P' of a different reference point R'. Note that the 2D distance of P' to R' is not minimal since P' is a 3D PCA. The point V denotes a general point on the trajectory. It is important to keep in mind that all points differ in their z-coordinate. Concerning the signs of the quantities, we have $d_0 < 0$, $\rho < 0$, $\varphi_P > 0$, and $\varphi_V > 0$ in this plot.

4.2 3D PCA

In this section, we will discuss an algorithm to find the 3D point of closest approach P' to a reference point R' given the Perigee parametrization of the track with respect to a different reference point R, see Fig. 4.2. In other words, one could say that we look for the 3D PCA (minimal $\sqrt{(d_0^2 + z_0^2)}$) of R' given the 2D PCA (minimal d_0) of R. Note that R' will often correspond to a vertex position estimate.

First, we express the distance between R' and an arbitrary point on the track V as a function of the azimuthal angle φ_V of the particle momentum at V. Then, we discuss how to minimize this function with respect to φ_V using Newton's method (see Ref. [2] for example). Let us start by expressing the x- and y-coordinates of the helix center O via the Perigee parameters at P. From geometrical considerations in Fig. 4.2, we find

$$x_O = x_R + |d_0| \sin\varphi_P - |\rho| \sin\varphi_P$$
$$y_O = y_R - |d_0| \cos\varphi_P + |\rho| \cos\varphi_P,$$

and, using the correct signs of d_0 and ρ,

$$x_O = x_R - (d_0 - \rho) \sin\varphi_P$$
$$y_O = y_R + (d_0 - \rho) \cos\varphi_P.$$

Note that these expressions are constant.

The x- and y-distance between an arbitrary point on the trajectory V and the reference point R' can then be found from geometrical considerations in Fig. 4.2. We find

$$\begin{aligned} d_x^2 &= (|\rho| \sin\varphi_V - (x_{R'} - x_O))^2 \\ &= (x_O - x_{R'} - \rho \sin\varphi_V)^2 \\ d_y^2 &= (|\rho| \cos\varphi_V - (y_O - y_{R'}))^2 \\ &= (y_{R'} - y_O - \rho\cos\varphi_V)^2 \\ &= (y_O - y_{R'} + \rho\cos\varphi_V)^2, \end{aligned}$$

where we know x_O and y_O thanks to the previous calculations. To obtain an expression for the z-distance, we first use the last equation of Eqs. 4.2 to find

$$z_V = z_P + v\cos\theta(t_V - t_P).$$

Remembering $z_P = z_R + z_0$ and using Eq. 4.13, we obtain

$$\begin{aligned} z_V &= z_P + v\cos\theta \frac{\rho}{v\sin\theta} (\varphi_P - \varphi_V + 2\pi(n_P - n_V)) \\ &= z_R + z_0 - \rho\cot\theta\,(\varphi_V - \varphi_P + 2\pi(n_V - n_P)) \,. \end{aligned} \tag{4.19}$$

The z-distance then reads

$$\begin{aligned} d_z^2 &= (z_V - z_{R'})^2 \\ &= (z_R + z_0 - z_{R'} - \rho\cot\theta\,(\varphi_V - \varphi_P + 2\pi(n_V - n_P)))^2 \,, \end{aligned}$$

and we can define a function whose minimization will yield the 3D PCA:

$$\begin{aligned} f(\varphi_V) &= \frac{1}{2}\left(d_x^2 + d_y^2 + d_z^2\right) \\ &= \frac{1}{2}\Big((x_O - x_{R'} - \rho\sin\varphi_V)^2 + (y_O - y_{R'} + \rho\cos\varphi_V)^2 \\ &\quad + (z_R + z_0 - z_{R'} - \rho\cot\theta\,(\varphi_V - \varphi_P + 2\pi(n_V - n_P)))^2\Big). \end{aligned}$$

To use the Newton method, we need to differentiate this function twice. We find:

$$\begin{aligned}\partial_{\varphi_V} f(\varphi_V) &= (x_O - x_{R'} - \rho \sin\varphi_V)(-\rho\cos\varphi_V)\\ &\quad + (y_O - y_{R'} + \rho\cos\varphi_V)(-\rho\sin\varphi_V)\\ &\quad + (z_R + z_0 - z_{R'} - \rho\cot\theta\,(\varphi_V - \varphi_P + 2\pi(n_V - n_P)))\,(-\rho\cot\theta)\\ &= (x_{R'} - x_O)\rho\cos\varphi_V + (y_{R'} - y_O)\rho\sin\varphi_V\\ &\quad + (z_{R'} - z_R - z_0 + \rho\cot\theta\,(\varphi_V - \varphi_P + 2\pi(n_V - n_P)))\,\rho\cot\theta,\end{aligned}$$

where we neglected all terms containing a Dirac delta distribution. Continuing to differentiate, we have

$$\begin{aligned}\partial^2_{\varphi_V} f(\varphi_V) &= -(x_{R'} - x_O)\rho\sin\varphi_V + (y_{R'} - y_O)\rho\cos\varphi_V\\ &\quad + \rho^2\cot^2\theta.\end{aligned}$$

Using the above formulae, one can employ the Newton method to find a *local* minimum of the 3D distance between the track and R'. It should be noted that we do not need to account for the number of times we passed the origin if we do not impose $\varphi_V \in [-\pi, \pi)$ during the optimization. As a consequence, we can drop the term $2\pi(n_V - n_P)$.

Finally, we obtain the 4D position of P' by geometrical considerations (for the x- and y-coordinate), Eq. 4.19 (for the z-coordinate) and Eq. 4.13 (for the time):

$$\begin{aligned}x_{P'} &= x_O + |\rho|\sin\varphi_{P'}\\ &= x_O - \rho\sin\varphi_{P'}\\ y_{P'} &= y_O - |\rho|\cos\varphi_{P'}\\ &= y_O + \rho\cos\varphi_{P'}\\ z_{P'} &= z_R + z_0 - \rho\cot\theta\,(\varphi_{P'} - \varphi_P + 2\pi(n_{P'} - n_P))\\ t_{P'} &= t_P - \frac{\rho}{v\sin\theta}\,(\varphi_{P'} - \varphi_P + 2\pi(n_{P'} - n_P))\,,\end{aligned}$$

where $\varphi_{P'} = \varphi_V^{\mathrm{opt}}$ is the value of φ_V after the optimization and we can drop the terms including $(n_{P'} - n_P)$ if we don't impose $\varphi_{P'} \in [-\pi, \pi)$.

List of Figures

Bibliography

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