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A Probability Model of a Pyramid Scheme

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Abstract

Periodically, the pyramid or "chain letter" scheme is offered to Americans under the guise of a business dealership. Recently, the FTC ordered Glen Turner's "Dare to be Great" firm to repay 44 million dollars to participants. In order to demonstrate that the potential gains are misrepresented by promoters, a probability model of the pyramid scheme is developed. The major implications are that the vast majority of participants have less than a ten percent chance of recouping their initial investment when a small profit is achieved as soon as they recruit three people and that, on the average, half of the participants will recruit no one else and lose all their money.

KEY WORDS: Statistics in the law; Probability model; Pyramid fraud; Probability bounds

1. Introduction

Periodically, the pyramid or "chain letter" scheme is offered to Americans under the guise of a business dealership. Recently, Glen Turner's Koscot Interplanetary Cosmetics firm has been charged with pyramiding by the FTC, SEC, and various state regulatory agencies [2]. The total loss to the public has been estimated to be 44 million dollars. The promoters offer people a dealership or sales job in which most of their remuneration comes from recruiting new dealers (or salespersons). The basic fraud underlying a typical pyramid scheme is that every participant cannot recruit enough other people to recoup his investment, much less make a profit, since the pool of potential participants is soon exhausted.

The usual method of prosecuting such schemes is to show that if the representation of the promotional brochures were valid (e.g., members could recruit two new people a month), then within a short period of time (about 18 months) the entire population of the United States would have to participate. Thus the last members would have no one to recruit. Although this argument based on geometric progression is sometimes rejected by courts as unrealistic [3], pyramid scheme operators have placed a quota (or limit) on the number of participants in a specific geographic area in order to evade this line of prosecution. This article develops a probability model of this quota-pyramid scheme, and derives the following results which also apply to unlimited schemes:

1. The vast majority of participants have less than a 10 percent chance of recouping their initial investment when a small profit is achieved as soon as three people are recruited.
2. On the average, half of the participants will recruit no one else and lose all their money.
3. On the average, about one-eighth of the participants will recruit three or more people.
4. Less than one percent of the participants can expect to recruit six or more new participants.

While these results can be approximately derived by ordinary limit theorems, for purposes of legal cases an absolute statement that a probability is small is more useful than an approximate statement. Thus the preceeding results are derived from a new probability bound on the sum of "small' binomial random variables (rv's) which is related to previous work of Hodges and LeCam [4].

2. Description of One Pyramid Scheme

A recent legal case in Connecticut [6] illustrates the confounding of legitimate business enterprise with a pyramid operation. People were offered dealerships in a "Golden Book of Values" for a fee of $2,500. In return for their investment dealers could earn money in two ways. In each geographic area dealers were to develop a Book of Values for eventual sale to the public. First, they were to sell advertisements to merchants for $195 apiece and could keep half as a commission. Each advertisement offered a product or service at a discount, so that a Book of Values containing 50 to 100 discount offers could be sold to the public. The public was to pay $15 for the Book of Values, of which dealers were to keep $12. Second, a dealer had the right to recruit other dealers and was to receive $900 for each new recruit. Since the creation of a complete Book of Values for sale to the public takes a substantial amount of time, clearly the recruitment of new dealers is the most lucrative aspect of the venture.

In the recruitment brochure the possibility of earning large sums of money was illustrated by the following example: A dealer will bring people to weekly "Opportunity Meetings" and should be able to enroll other dealers at the rate of two per month. Thus at the end of one year, the participant should receive $21,600 from the recruitment aspect alone. The prosecution showed that this misrepresents the earnings potential by asking the following question: "Suppose dealers who are enrolled can enroll two other dealers per month; as time went by, what would happen?" Professor Margolin (of Yale) testified that there would be a tripling of the number of dealers per month and by the end of 18 months, the geometric progression would exhaust the population of the United States. Clearly, the cited recruitment brochure is misleading as all participants cannot come close to earning the indicated amount of money.

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The Golden Book of Values pyramid system had an extra statistical nuance; i.e., there was a quota of 270 dealerships for the State of Connecticut. The Court noted that if each new dealer was successful in recruiting two dealers per month, only 27 would make a profit and the other 243 would lose money depending on how far down the pyramid they were. Since a real pyramid operation would not be as regular as the Court described it, i.e., even at the beginning every participant would not enroll exactly two new dealers each month, in the next section we develop a probability model of the pyramid scheme. The model enables us to calculate the probability distribution of the number of people each participant will recruit and realize how strongly the probability of recouping one's initial investment depends on *when* the participant enters the pyramid scheme. Furthermore, the fraction of participants who can expect to recruit no one can be derived.

3. Calculating the Expected Return and Probability of Earning a Profit for Individual Participants in a Quota Pyramid System

Economists evaluate the profitability of a business venture by comparing the initial investment to the expected return over a period of time. Suppose one is offered the opportunity to pay c dollars to enter a pyramid scheme which will terminate when the total number of participants is N, where the fee for finding a new recruit is d dollars. Should one join? The answer is yes only if the expected number of people one will recruit, say R, is greater than c/d, i.e., one's expected earnings (Rd) are larger than the cost (c) of entering the plan. In this section we calculate the expected number of people the kth participant will recruit assuming that all current participants have the same chance of recruiting the next member.

For ease in exposition we focus on the kth entrant into the system. Since there are now k participants, each of whom presumably is recruiting, the probability that any particular one of the k current members recruits the next one is $1/k$. Once the $k+1$st participant is recruited, each member has a chance of $1/(k+1)$ of recruiting the $k+2$nd participant, etc. Thus the number of people the kth participant will recruit is expressible as the sum of independent binomial rv's,

$$S_k = \sum_{i=k}^{N-1} X_i, \tag{3.1}$$

where

$$X_i = 1, \quad \text{with probability } p_i = 1/i$$
$$= 0, \quad \text{with probability } 1 - 1/i.$$

Thus the expected number of people the kth person will recruit equals

$$\sum_{i=k}^{N-1} 1/i \sim \ln[(N - \tfrac{1}{2})/(k - \tfrac{1}{2})]. \tag{3.2}$$

An immediate consequence of (3.2) is that once k is $\geq N/e$, or about $.37N$, any future participant can expect to recruit no more than *one* person. Thus only the 37 percent who join first can expect to recruit at least *one* new participant.

Another approach to demonstrating that a participant who joins the scheme after its initial phase has a *small* chance of recouping their investment is to calculate the probability that they will recruit the minimum number of people, $b = [c/d] + 1$, to achieve this. In our illustrative example, this value is 3. In order to compute $P(S_k \geq 3)$, statisticians use the Poisson approximation to the sum of binomials (3.1), as the p_i are small and decrease to zero. In the Appendix we describe a method of approximating S_k by Poisson rv P_k which is stochastically larger than S_k, and the probabilities presented in Table 1 are derived from these results and are, therefore, upper bounds for the actual probabilities.

TABLE 1
The Expected Number of People Each Participant will Recruit and Upper Bounds for the Probability of Recruiting at Least 2 or 3 New Members (N = 270)

Position of Entry (k)	Expected No. of Recruits	Probability of Recruiting at least r New Members: r = 2	r = 3
5	4.208	.9226	.7909
10	3.398	.8529	.6598
20	2.6500	.7422	.4941
30	2.227	.6521	.3846
40	1.931	.5750	.3047
50	1.703	.5077	.2435
60	1.517	.4479	.1955
75	1.291	.3699	.1407
90	1.106	.3032	.1008
100	1.1000	.2641	.0802
120	.8160	.1968	.0497
135	.697	.1547	.0338
150	.591	.1189	.0222
180	.407	.0635	.0083
210	.2524	.0270	.0022
240	.1182	.0065	.0003

The results in Table 1 show that once a quota pyramid reaches *one-third* of its limit the probability a new member will regain his investment is less than ten percent.

4. The Expected Return to All Participants

In Section 3 we were concerned with the probability of each individual recruiting enough future members to regain the entrance fee. We now demonstrate that pyramid-scheme investors are defrauded as a class.

The simplest proof of this is to notice that at any stage of the process (say, K people are enrolled), the promoter (the first person) has received $(K - 1)c$ and has paid out $(K - 2)d$. Hence the promoter has a net profit of

$$c + (K - 2)(c - d),$$

and the fraction of investment that has been returned to the participants is

$$[(K - 2)/(K - 1)]\cdot(d/c).$$

Thus the portion of all invested dollars returned to the participants is slightly less than d/c, that is, the ratio of the fee earned for recruiting one new member to the initial investment. In the actual case used for illustration, this is only .36. Thus, as a class, participants will lose 64 percent of their investment.

Another interesting consequence of the probability model is that on the average about *half* of the participants will recruit nobody and will lose their whole investment. This can be seen by noting that the probability that the kth entrant will fail to recruit anyone is

$$P_k(0) = \prod_{i=k}^{N-1} (1 - 1/i) = (k - 1)/(N - 1).$$

Thus the expected number of participants who are "shut out" is

$$\sum_{2}^{N} P_k(0) = \frac{1}{(N - 1)}(1 + \ldots + N - 1) = \frac{N}{2},$$

i.e., half of the investors will lose everything they paid to join the system. Moreover, this remains true for any value of d (the amount paid for enrolling a new member). Thus even if *all* the money paid in were returned to investors, *half of them can expect to receive nothing*.

One might question the relevance of the previous result in the context of a fraud case if a significant fraction of the participants were big winners. When we replace the rv S_k denoting the number of people the kth entrant recruits by its Poisson majorizer P_k, one can show (see Appendix) that as $N \to \infty$, the *proportion* of the participants who recruit exactly r people approaches $2^{-(r+1)}$ so that the fraction who recruit at least r is 2^{-r}. Thus only one-eighth of the participants can expect to recruit at least three members, and only one in 16 million can expect to recruit 24 or more people. Thus our model agrees with the findings of Judge Naruk in the case described when he noted that no one had earned an amount of money near that claimed in the brochure.

In light of this and other facts, Judge Naruk permanently enjoined the defendants from selling or authorizing others to sell Golden Book dealerships and from instituting any other multi-level merchandising plan in Connecticut without express court approval.

Appendix: A Probability Bound for the Sum of Poisson-Binomial Variates

Let X_i, $i = 1, \ldots, n$, be independent binomial rv's with $p_i = P(X_i = 1)$, and let $S = \Sigma X_i$. When the probabilities p_i are small, we desire a tight upper bound rather than an approximation to

$$P\left\{\sum_{i=1}^{n} X_i > a\right\}, \tag{A.1}$$

where a is a specified integer usually greater than the expected value, Σp_i, of the rv's.

In order to derive a bound for (A.1), we introduce Poisson rv's Y_i, which are stochastically larger than the X_i's, i.e., we choose the parameter λ_i of Y_i to satisfy

$$P(Y_i = 0) = P(X_i = 0) = 1 - p_i, \tag{A.2}$$

i.e.,

$$e^{-\lambda_i} = 1 - p_i \text{ or } \lambda_i = -\ln(1 - p_i). \tag{A.3}$$

In order to give X_i and Y_i a bona fide joint distribution, following Hodges and LeCam, we define

$$P(X_i = 0,\ Y_i = 0) = 1 - p_i,$$

and

$$P(X_i = 1,\ Y_i = k) = e^{-\lambda_i}\lambda_i^k/k!,$$

where λ_i and p_i obey (A.3).

As $X_i \leq Y_i$ for each i, $\Sigma X_i \leq \Sigma Y_i$ and

$$P(\Sigma X_i \geq a) \leqq P(\Sigma Y_i \geq a). \tag{A.5}$$

As ΣY_i has a Poisson distribution, the probability on the right is readily computable once λ_i is expressed in terms of p_i. From the Taylor expansion,

$$-\ln(1 - x) = \sum_{1}^{\infty} x^j/j,$$

it follows that

$$\sum_{j=1}^{k} \frac{x^j}{j} \leq -\ln(1 - x) \leq \sum_{j=1}^{k-1} \frac{x^j}{j} + \frac{x^k}{k}\frac{1}{(1 - x)}, \tag{A.6}$$

so each λ_i can be obtained to any desired accuracy. For practical purposes, the choice of $k = 3$ usually suffices, so (A.6) becomes

$$p_i + p_i^2/2 + p_i^3/3 \leq \lambda_i \leq p_i + p_i^2/2 + (p_i^3/3)[1/(1 - p_i)].$$

When the $\{p_i\}$ decrease, the difference between the bounds on the parameter $\Sigma_j^N \lambda_i$ of the Poisson rv majorizing S is

$$[(1 - p_j)^{-1} - 1]\left[\sum_{i=j}^{N} p_i^3\right]\Big/3.$$

Before applying the above method to our special case we present the analog of Hodges and LeCam's results for the difference between $P(S \geq a)$ and our approximation P_λ. Specifically, we have

Lemma: For any constant a,

$$P(P_\lambda \geq a) - P(S \geq a) \leq \sum p_i^2. \tag{A.7}$$

Proof: For each i,

$$\begin{aligned} P(Y_i > X_i) &= P(Y_i \neq X_i) \\ &= \sum_{k=2}^{\infty} e^{-\lambda_i}\frac{\lambda_i^k}{k!} \\ &= 1 - e^{-\lambda_i} - \lambda_i e^{-\lambda_i} \\ &= p_i + (1 - p_i)\ln(1 - p_i). \end{aligned}$$

As $e^{-x} \geq 1 - x$, $-x \geq \ln(1 - x)$, so

$$p_i + (1 - p_i)\ln(1 - p_i) \leq p_i - (1 - p_i)p_i = p_i^2.$$

By Boole's inequality, $P(P_\lambda > S) \leq \Sigma p_i^2$.

Application to the Pyramid Scheme

In our example, $p_i = 1/i$, and we desire to approximate

$$S_k = \sum_{i=k}^{N=1} X_i$$

by a Poisson variable.

In our case we can obtain an explicit expression for γ_k rather than using a Taylor-series development as

$$\lambda_i = -\ln(1 - 1/i) = \ln(i/i - 1). \qquad (A.8)$$

Thus

$$\gamma_k = \sum_{i=k}^{N-1} \lambda_i = \sum_{k}^{N-1} [\ln i - \ln(i - 1)] = \ln\frac{(N-1)}{(K-1)},$$

so that

$$P(S_k \geq r) \leq \sum_{i=r}^{\infty} e^{-\gamma_k} \frac{\gamma_k^{\,i}}{i!} = 1 - \sum_{i=1}^{r-1} \frac{(k-1)}{(N-1)} \frac{\gamma_k^{\,i}}{i!}. \qquad (A.9)$$

TABLE 2

Comparison of Our Bounds to the Exact Values (N = 270)

Index (k)	Exact $P(S_k \geq 2)$	UB for $P(S_k \geq 2)$	Exact $P(S_k \geq 3)$	UB for $P(S_k \geq 3)$
5	.92064	.92255	.78289	.79087
10	.85104	.85287	.65358	.65978
25	.69346	.69517	.43053	.43464
40	.57344	.57504	.30162	.30470
50	.50613	.50765	.24095	.24354
75	.36851	.36986	.13900	.14074
90	.30195	.30319	.09943	.10081
135	.15378	.15472	.03311	.03376
180	.06290	.06353	.00807	.00833
210	.02655	.02696	.00212	.00222
240	.00626	.00646	.00023	.00025

Note: The ordinary Poisson approximation also yields precise estimates of the exact probabilities. These estimates are slightly *low* for the early values of k and become slightly *high* for larger values of k as would be expected from the literature [1] on the Poisson approximation to the sum of identically and independently distributed (iid) binomial rv's.

Since S_k is the sum of nonidentically distributed binomial rv's, a compact formula for its exact distribution is not available and a computer is needed. In Table 2, we compare the exact value of $P(S_k \geq 2)$ and $P(S_k \geq 3)$ to the bounds we obtain from (A.9). Clearly the bounds are quite close.

As our rv's P_k approximating S_k are so close, we can derive an accurate approximation to the expected fraction of participants who will recruit at least r people. Formally, we have

Theorem: Let $X_2, X_3, \ldots, X_N$ be a sequence of Poisson rv's with parameters $\gamma_k = [\ln(N - 1/k - 1)]$. Then

$$\frac{1}{(N-1)} \sum_{k=2}^{N-1} P(X_k = r) \rightarrow 2^{-(r+1)}, \quad r = 0, 1, 2, \ldots, \qquad (A.10)$$

as $N \rightarrow \infty$.

Proof: As $P(X_k = r) = \gamma_k^{\,r} e^{-\gamma_k}/r!$, the left side of (A.10) is

$$\frac{1}{r!}(N-1)^{-1} \sum_{2}^{N-1} \frac{(k-1)}{(N-1)} \left[\ln\left(\frac{N-1}{k-1}\right)\right]^r. \qquad (A.11)$$

Letting $v = (k - 1)/(N - 1)$, (A.11) is a Riemann approximation to

$$(r!)^{-1} \int_0^1 v\left[\ln\left(\frac{1}{v}\right)\right]^r dv = (r!)^{-1} \int_0^\infty z^r e^{-2z} dz = 2^{-(r+1)}.$$

Hence for large N the expected fraction of all participants who recruit at least r people is $1/2^r$ for $r = 0, 1, 2, \ldots$.

In order to see how fast the limit is approached we computed the exact values of (A.10) when N = 270 and 1,000 for r = 1, 2, and 3. The resulting values which have limits $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$, were .24991, .12475, and .06202 (N = 270) and .24999, .12497, and .06244 (N = 1,000).

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